AN INTELLIGENT METHOD FOR ACCELERATING THE CONVERGENCE OF DIFFERENT VERSIONS OF SGMRES ALGORITHM

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ABSTRACT

In a wide range of applications, solving the linear system of equations Ax = b is appeared. One of the best methods to solve the large sparse asymmetric linear systems is the simplified generalized minimal residual (SGMRES(m)) method. Also, some improved versions of SGMRES(m) exist: SGMRES-E(m, k) and SGMRES-DR(m, k). In this paper, an intelligent heuristic method for accelerating the convergence of three methods SGMRES(m), SGMRES-E(m, k), and SGMRES-DR(m, k) is proposed. The numerical results obtained from implementation of the proposed approach on several University of Florida standard matrixes confirm the efficiency of the proposed method.

Keywords

Artificial Intelligence, Heuristic Algorithms, Linear systems of equations, SGMRES.

1. INTRODUCTION

In a wide range of applied sciences, solution of problems is obtained from solving a linear system of equations Ax = b. In numerical analysis, the algorithms for solving linear systems commonly use one of the two following methods: Direct methods, Iterative methods.

According to the coefficient of linear devices and matrix A, we can use one of these methods. If the matrix dimension is too small, direct methods such as Gaussian elimination method, Gauss-Jordan and LU decomposition is preferred. These methods are composed of a finite number of steps. On the other hand, iterative methods are based on calculating a sequence of approximations to find the solution. In these methods, to stop the algorithm either an accurate solution is found or a certain number of iterations is performed. If the matrix A is relatively large, triangular-based direct methods are not recommended, because this method requires plenty of time and storage space. In addition, in many cases the coefficient matrix is sparse and triangular process destroys the sparseness of matrix, such we are faced with a large dense matrix. To solve such problems, it is recommended to use iterative methods which do not change the sparseness nature of matrix A.

One of the main iterative methods, which nowadays has found many applications, is the generalized minimal residual method (GMRES) proposed by Saad and Schultz [1]. Simpler GMRES (SGMRES) method is a new version of GMRES, which was proposed by Walker and Zhou [2] and analyzed by Jiránek et al. [4]. SGMRES(m)) is a restarted version of SGMRES which restarts at the iteration when the Krylov subspace reaches dimension m, then the current solution is used as the new initial guess for the next m iterations.

Despite the good performance of these methods, it seems that we can make changes in their structure to accelerate their convergence. In this study, an intelligent heuristic method to change the update equations of three methods SGMRES(m), SGMRES-E(m, k), and SGMRES-DR(m, k) is proposed. The experimental results on several University of Florida standard matrixes confirm the efficiency of the proposed method.

2. THE SGMRES AND ITS VARIANTS

The generalized minimal residual (GMRES) method developed by Saad and Schultz [1] is one of the most popular iterative methods for solving the large sparse nonsymmetrical system of linear equations Ax=b, in which $A \in R^{n \times n}$ is nonsingular, $b \in R^n$ is a right-hand side vector, and $x \in R^n$ is the sought-after solution. Let $x_0 \in R^n$ be the initial guess, and $r_0 = b - Ax_0$ the initial residual vector. At the mth iteration step, GMRES finds the approximate solution x_m in the affine subspace $x_0 + K_m(A, r_0)$, such that x_m minimizes the Euclidean norm of the residual, i.e.:

$$\|r_m\| = \|b - Ax_m\| = \|b - A(x_0 + z_m)\| \\ = \min_{z \in K_m(A, r_0)} \|b - A(x_0 + z)\| = \min_{z \in K_m(A, r_0)} \|r_0 - Az)\| \quad (1)$$

Note that $K_m(A, r_0) = span\{r_0, Ar_0, \dots, A^{m-1}r_0\}$ is the mth Krylov subspace constructed by the matrix A and the initial residual vector r_0. It is obvious that (1) is corresponding to the orthogonality condition, i.e.:

$$r_m \perp AK_m(A, r_0) \quad (2)$$

where the orthogonality relation is established upon the Euclidean inner product.
The traditional implementation of GMRES is inspired from the Arnoldi process [3]. Application of m steps of the Arnoldi process to the matrix A with the nonzero residual vector r_0 yields the Arnoldi factorization

$$AV_m = V_{m+1}\widehat{H}_m$$

where the columns of the matrix V_m form the orthonormal basis of the Krylov subspace $K_m(A, r0)$ and $\widehat{H}_m \in R^{(m+1) \times m}$ is an upper Hessenberg matrix. We can reformulate the least-squares problem as the reduced minimization problem

$$\|b - Ax_m\| = \|b - A(x_0 + V_m y_m)\| = \min_{y \in R^m} \|\beta e_1 - \widehat{H}_m y\|$$

where $\beta = \|r_0\|$ and e_1 is the first column of an identity matrix of order $m + 1$. The QR factorization of $\widehat{H}_m$ with Givens rotations can be used to solve the reduced minimization problem.

Simpler GMRES (SGMRES) method is a new version of GMRES, which was proposed by Walker and Zhou [2] and analyzed by Jiránek et al. [4]. In their method, instead of building an orthonormal basis of $K_m(A, r_0)$, an orthonormal basis V_m of $AK_m(A, r_0)$ is established and then carries out the orthogonality relation (2). In a specific manner, suppose $Z_m = [z_1, z_2, . . ., z_m]$ be a basis of $K_m(A, r_0)$. Then, the QR factorization of AZ_m can be used to acquire the orthonormal basis V_m of $AK_m(A, r_0)$, i.e.

$$AZ_m = V_m R_m$$

where $V_m = [v_1, v_2, . . . , v_m]$ has orthonormal columns and R_m is an m×m upper triangular and nonsingular matrix. By accomplishing the orthogonality relation (2), we can compute the mth residual vector r_m recursively as

$$r_m = r_0 - V_m(V_m^T r_0) = r_{m-1} - \alpha_m v_m \quad m \geq 1$$

where $\alpha_m = v_m^T r_0$. The corresponding approximate solution is

$$x_m = x_0 + Z_m t_m$$

where t_m is the solution of the upper triangular system

$$R_m t_m = [\alpha_1, \alpha_2, ..., \alpha_m]^T.$$

By lapse of iterations, the amount of required computational time and space for GMRES or simpler GMRES increases meaningfully; and this subject makes GMRES and simpler GMRES impractical. In order to overcome this shortcoming, the restarted version of these algorithms is proposed. In restarted GMRES (GMRES(m)) or restarted simpler GMRES (SGMRES(m)), GMRES or SGMRES is restarted at the moment that the Krylov subspace reaches dimension m, then the current solution is used as the new initial guess for the next m iterations. Some details of SGMRES(m) is presented in Algorithm 1[13].

Algorithm 1 (SGMRES(m)).

Input: A : the coefficient matrix; b : the right-hand side; x_0 : an initial approximation; m : the maximal dimension of the Krylov subspace; ε : a tolerance.

Output: An approximate solution x.

1. Compute $r_0 = b - Ax_0$, $z_1 = r_0/\|r_0\|$, $\hat{z}_1 = Az_1$, $r_{11} = \|\hat{z}_1\|$, $v_1 = \hat{z}_1/r_{11}$
2. Compute $\alpha_1 = v_1^T r_0$, $r_1 = r_0 - \alpha_1 v_1$
3. For $j = 2,3, ..., m$
   3.1. $z_j = v_{j-1}$
   3.2. $\hat{z}_j = Az_j$
   3.3. For $i = 1,2, ..., j-1$
   $$r_{ij} = v_i^T \hat{z}_j$$
   $$\hat{z}_j = \hat{z}_j - r_{ij} v_i$$
   End For
   3.4. $r_{ij} = \|\hat{z}_j\|$, $v_j = \hat{z}_j/r_{jj}$, $\alpha_j = v_j^T r_{j-1}$

3.5. $r_j = r_{j-1} - \alpha_j v_j$

3.6. If $\|r_j\| < \varepsilon$ then solve the linear triangular system

$$R_j t = \alpha$$

with $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_j]^T$ and form the approximate solution $x_m = x_0 + Z_m t$ and stop

4. Solve the linear triangular system

$$R_j t = \alpha$$

With $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_j]^T$ and form the approximate solution $x_m = x_0 + Z_m t$

5. Set $x_0 = x_m$ and go to Step 1.

This implementation of SGMRES(m) was proposed by Walker and Zhou [2]. In their implementation, they used $Z_m = [r_0/\|r_0\|, V_{m-1}]$. Note that different restarted simpler GMRES, called the residual-based restarted simpler GMRES (RB-SGMRES(m)), is proposed in [4]. RB-SGMRES(m) uses

$$Z_m = [r_0/\|r_0\|, r_1/\|r_1\|, \ldots, r_{m-1}/\|r_{m-1}\|]$$

and is equivalent to SGMRES(m). The implementation of RB-SGMRES(m) is closely similar to that of SGMRES(m) except that $z_j = r_{j-1}/\|r_{j-1}\|$ in Step 3.1 in Algorithm 1. In [4], it has been shown that SGMRES(m) is essentially unstable, but RB-SGMRES(m) is conditionally stable provided that we have some reasonable residual decrease.

In general, because the Krylov subspace dimension is restricted at each cycle for the restarted methods, often, the convergence for the matrix with A having small eigenvalues slows down. The main reason of this is that the Krylov subspace used at each cycle does not include good approximations to the eigenvectors corresponding to small eigenvalues. To accelerate the convergence of restarted GMRES, it is proposed to compute the approximate eigenvectors corresponding to small eigenvalues in modulus at each restart, and then augment the Krylov subspace by these approximate eigenvectors to improve the convergence of restarted GMRES. This kind of methods includes GMRES-E [5], GMRES-IR [6], and GMRES-DR [7]. These methods are equivalent at the end of each cycle. Although, GMRES-IR needs less matrix-vector products than GMRES-E at each cycle, it is slightly more complicated to implement. Moreover, GMRES-IR suffers from stability problems because it uses the implicitly restarted Arnoldi process [8], which may suffer from stability when a shift used in the implicitly restarted Arnoldi process is close to a true eigenvalue of the problem. GMRES-DR which is a good improvement of GMRES-IR has the efficiency of GMRES-IR, but is simpler to use and does not have the numerical stability problems [7].

Boojhawon and Bhuruth[9] by using augmentation technique applied the idea in GMRES-E to SGMRES, and proposed the simpler GMRES method augmented with approximate eigenvectors (SGMRES-E). The main advantage of SGMRES-E over GMRES-E is SGMRES-E requires a less computational time than GMRES-E.

Ref [13] improves the SGMRES-E by following the idea behind GMRES-DR and proposed the simpler GMRES method with deflated restarting (SGMRES-DR). SGMRES-DR includes the harmonic Ritz vectors corresponding to the smallest eigenvalues of the coefficient matrix A at the

beginning of each cycle. SGMRES-DR requires less matrix-vector products than SGMRES-E, and thus is more efficient.

2.1 SGMRES with deflated restarting

Suppose $W \subset R_n$ be a subspace with dimension m, the columns of $Z_m = [z_1, z_2, \ldots, z_m]$ be a basis of W, $x_0 \in R_n$ be the initial guess, and $r_0 = b - Ax_0$ the initial residual vector. The purpose is to find the approximate solution x_m in the affine subspace $x_0 + W$ by imposing the minimization of the residual norm. It is equivalent to finding x_m such that

$$r_m \perp AW \qquad (3)$$

where $r_m = b - Ax_m$, and $x_m = x_0 + Z_m y_m$ for some $y_m \in R_m$.

By applying the QR decomposition of AZ_m we have:

$$AZ_m = V_m R_m \qquad (4)$$

and applying the orthogonality relation (3), the expression of the mth residual vector is obtained:

$$r_m = r_0 - V_m V_m^T r_0 = r_{m-1} - v_m (v_m^T r_0)$$

The harmonic Ritz vectors corresponding to the smallest eigenvalues of A is used in SGMRES-DR because they are better approximate eigenvectors for eigenvalues with small modulus [10]. Note that the term "the smallest eigenvalues" means the smallest eigenvalues in modulus. The harmonic Ritz vector $u = Z_m y$ satisfies the following orthogonality condition

$$(Au - \lambda u) \perp AZ_m$$

which is equivalent to

$$V_m^T (AZ_m y - \lambda Z_m y) = 0$$

i.e.,

$$R_m y = \lambda V_m^T Z_m y \qquad (5)$$

Suppose $y_1, y_2, \ldots, y_k$ are k eigenvectors corresponding to k smallest eigenvalues of the reduced generalized eigenvalue problem (5).

In the SGMRES-E, proposed by Boojhawon and Bhuruth in [9], approximate eigenvectors used to augment the Krylov subspaces are harmonic Ritz vectors $\hat{x}_j = Z_m y_j, j = 1, 2, \ldots, k$. The implementation of SGMRES-E is outlined in Algorithm 2 [13].

Algorithm 2 (SGMRES-E(m, k)).

Input: A : the coefficient matrix; b : the right-hand side; x_0 : an initial approximation; m : the maximal dimension of the Krylov subspace; k : the number of harmonic Ritz vectors; ε : a tolerance.
Output: An approximate solution x.

1. Apply one cycle of SGMRES(m) to generate Z_m, V_m, R_m, x_m, and r_m.
2. Compute the eigenvalues and eigenvectors of the generalized eigenvalue problem (6) by the QZ algorithm [11]. Let y_1, y_2, . . .,y_k be k eigenvectors corresponding to k smallest eigenvalues of (6).
3. Form k harmonic Ritz vectors $\hat{x}_j = Z_m y_j$, $j = 1,2,\dots,k$
4. Set $x0 = x_m$, and compute $r_0 = r_m$, $z_1 = r_0/\|r_0\|$, $\hat{z}_1 = Az_1$, $r_{11} = \|\hat{z}_1\|$, $v_1 = \hat{z}_1/r_{11}$
5. Compute $\alpha_1 = v_1^T r_0$, $r_1 = r_0 - \alpha_1 v_1$
6. For $j = 2,3,\dots,m$
   6.1. If $j \leq m - k$ then $z_j = v_{j-1}$ else $z_j = \hat{x}_{j-m+k}$
   6.2. $\hat{z}_j = Az_j$
   6.3. For $i = 1,2,\dots,j-1$
   $r_{ij} = v_i^T \hat{z}_j \hat{z}_j = \hat{z}_j - r_{ij} v_i$
   End For
   6.4. $r_{jj} = \|\hat{z}_j\|$, $v_j = \hat{z}_j/r_{jj}$, $\alpha_j = v_j^T r_{j-1}$
   6.5. $r_j = r_{j-1} - \alpha_j v_j$
   6.6. If $\|r_j\| < \varepsilon$ then solve the linear triangular system
   $$R_j t = \alpha$$
   with $\alpha = [0,\dots,0,\alpha_{k+1},\dots,\alpha_j]^T$ ‹ and form the approximate solution $x = x_0 + Z_j t$, and stop.
7. Solve the linear triangular system $R_m t = \alpha$ with $\alpha = [\alpha_1,\alpha_2,\dots,\alpha_j]^T$ and form the approximate solution $x_m = x_0 + Z_m t$. Go to Step 2.

Suppose $Y_k = [y_1, y_2, \dots, y_k]$ and let $P_k L_k = Y_k$ be the QR decomposition of Y_k. Multiplying (4) by P_k from the right yields

$$AZ_m P_k = V_m R_m P_k$$

Let $Q_k R_k^{new} = R_m P_k$ be the QR decomposition of $R_m P_k$. From the above equality, we obtain

$$AZ_m P_k = V_m Q_k R_k^{new}$$

Define

$$Z_k^{new} = Z_m P_k , \qquad V_k^{new} = V_m Q_k$$

So, we have

$$AZ_k^{new} = V_k^{new} R_k^{new}$$

where $Z_k^{new}, V_k^{new} \in R^{n \times k}$, and V_k^{new} has orthonormal columns, $R_k^{new} \in R^{k \times k}$ is an upper triangular matrix. This is a QR decomposition with k new vectors. We now restart the GMRES method. Note that

$$x_0^{new} = x_m, \ r_0^{new} = r_m = r_0 - V_m V_m^T r_0$$

Therefore, the kth residual vector at the new cycle is

$$r_k^{new} = r_0^{new} - V_k^{new}\left((V_k^{new})^T r_0^{new}\right)$$

Since

$$(V_k^{new})^T r_0^{new} = Q_k^T V_m^T r_m = Q_k^T V_m^T (r_0 - V_m V_m^T r_0) = 0$$

we obtain

$$r_k^{new} = r_0^{new}$$

to extend the new QR factorization of order *k* (7) to obtain a QR factorization of order *m*, i.e., a QR factorization with *m* vectors.

In SGMRES-DR, we have

$$z_{j+1}^{new} . j = k, k+1, \ldots, m-1 \, as \, z_{j+1}^{new} = v_j^{new}$$

Then, the expansion of (7) to a QR factorization of order m can be done as follows: the vector $Az_{k+1}^{new} = Av_k^{new}$ is orthogonalized against V_k^{new} and normalized to give v_{k+1}^{new}, after which R_{k+1}^{new} is formed from R_k^{new}. With $z_{j+1}^{new} = v_j^{new}, j = k, k+1, \ldots, m-1$, the process is iterated until a QR factorization of order m is obtained. The implementation of SGMRES-DR is outlined in Algorithm 3[13].

Algorithm 3 (SGMRES-DR(m, k)).

Input: A : the coefficient matrix; b : the right-hand side; x_0 : an initial approximation; m : the maximal dimension of the Krylov subspace; k : the number of harmonic Ritz vectors; ε : a tolerance.
Output: An approximate solution x.

1. Apply one cycle of SGMRES(m) to generate Z_m, V_m, R_m, x_m, and r_m.
2. Compute the eigenvalues and eigenvectors of the generalized eigenvalue problem (6) by the QZ algorithm [11]. Set $Y_k = [y_1, y_2, \ldots, y_k]$, where $y_1, y_2, \ldots, y_k$ are k eigenvectors corresponding to k smallest eigenvalues of (6).
3. Compute the QR decomposition of Y_k: $P_k L_k = Y_k$ and the QR decomposition of $R_m P_k$: $Q_k R_k^{new} = R_m P_k$.
4. Set $Z_k^{new} = Z_m P_k$ and $V_k^{new} = V_m Q_k$.
5. Let $Z_k = Z_k^{new}, V_k = V_k^{new}, R_k = R_k^{new}, x_0 = x_m, r_0 = r_m, r_k = r_0$
6. For $j = k+1, k+2, \ldots, m$
    6.1. $z_j = v_{j-1}$
    6.2. $\hat{z}_j = Az_j$
    6.3. For $i = 1, 2, \ldots, j-1$
    $r_{ij} = v_i^T \hat{z}_j \hat{z}_j = \hat{z}_j - r_{ij} v_i$
    End For
    6.4. $r_{jj} = \|\hat{z}_j\|, v_j = \hat{z}_j / r_{jj}, \alpha_j = v_j^T r_{j-1}$
    6.5. $r_j = r_{j-1} - \alpha_j v_j$
    6.6. If $\|r_j\| < \varepsilon$, then solve the linear triangular system
    $R_j t = \alpha$

With $\alpha = [0, \ldots, 0, \alpha_{k+1}, \ldots, \alpha_m]^T$, and form the approximate solution $x_m = x_0 + Z_m t$.
Go to Step 2.

If the eigenvector y_i at Step 2 in Algorithm 3 is complex, the matrix Y_k should include both the real and imaginary parts of y_i. In this situation, we may have to tune k eigenvectors at a restart.

After running one cycle of SGMRES(m), we reach to $Z_m = [r_0/\|r_0\|, V_{m-1}]$. Based on [12], the matrix $V_m^T Z_m$ in the generalized eigenvalue problem (5) can be formulated as

$$
V_m^T Z_m = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix} [r_0/\|r_0\| \quad V_{m-1}] = \begin{bmatrix} \dfrac{v_1^T r_0}{\|r_0\|} & 1 & & & \\ \dfrac{v_2^T r_0}{\|r_0\|} & 0 & 1 & & \\ \vdots & & 0 & \ddots & \\ & & & \ddots & 1 \\ \dfrac{v_m^T r_0}{\|r_0\|} & & & & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{v_1^T r_0}{\|r_0\|} & 1 & & & \\ \dfrac{v_2^T r_1}{\|r_0\|} & 0 & 1 & & \\ \vdots & & 0 & \ddots & \\ & & & \ddots & 1 \\ \dfrac{v_m^T r_{m-1}}{\|r_0\|} & & & & 0 \end{bmatrix}
$$

Here, the next identity: for all $j > i, v_j^T r_i = v_j^T r_{j-1}$, has been used. Because $v_j^T r_{j-1}, j = 1,2,\dots m$, has been generated in applying one cycle of SGMRES (m), the matrix $V_m^T Z_m$ without additional inner products of vectors can be formed with order n.

For other cycles, we have $\quad Z_m = [z_1, \dots, z_k, v_k, \dots, v_{m-1}]$

$$
\text{So,} \quad V_m^T Z_m = \begin{bmatrix} v_1^T \\ \vdots \\ v_k^T \\ v_{k+1}^T \\ \vdots \\ v_m^T \end{bmatrix} [z_1, \dots, z_k, v_k, \dots, v_{m-1}] = \begin{bmatrix} v_1^T z_1 & \cdots & v_1^T z_k & & & & \\ \vdots & \vdots & \vdots & & & & \\ v_k^T z_1 & \vdots & v_k^T z_k & 1 & & & \\ v_{k+1}^T z_1 & \vdots & v_{k+1}^T z_k & 0 & 1 & & \\ v_{k+2}^T z_1 & \vdots & v_{k+2}^T z_k & & 0 & \ddots & \\ \vdots & \vdots & \vdots & & & \ddots & 1 \\ v_m^T z_1 & \cdots & v_m^T z_k & & & & 0 \end{bmatrix}
$$

Thus, km additional inner products of vectors with order n to form $V_m^T Z_m$ is required.

3. THE PROPOSED METHOD

As mentioned in the previous section, methods for solving linear system Ax = b start with an initial vector x, and iteratively try to change the values of these vectors in order to reduce the estimation error. In this methods, unfortunately, the process of calculating the amount of change of the vector x is a time consuming process. Therefore, in this paper, we will use an intelligent heuristic method to quickly predict the amount of change of the vector x in each iteration. The proposed heuristic is as follows: "The value of each dimension d of vector x which in c previous iterations of the algorithm steadily decreases (increases) is likely to decrease (increase) further in the next iteration." So, without much calculation, we can perform intelligent changes in the direction of each dimension of vector x. By this assumption, in practice dramatic convergence will be reached.

By changing the step 6 of algorithm 1, the Improved SGMRES(m) (ISGMRES(m)) is obtained. The implementation of ISGMRES(m) is outlined in Algorithm 4.

Algorithm 4: The template of ISGMRES(m)
1. Step 1 to step 5 of algorithm 1.
2. Decrease (increase) the value of each dimension d of vector x which in c previous iterations of the algorithm steadily decrease (increase), by $\beta * x'(d)$, where β is a real number that is called the update rate and $x'(d)$ is the amount of change in the value of vector x in dimension d in the before iteration.
3. Goto step 1.

By changing the step 8 of algorithm 2, the Improved SGMRES-E(m,k) (ISGMRES-E(m,k)) can be obtained. The implementation of ISGMRES-E(m,k) is outlined in Algorithm 5.

Algorithm 5: The template of ISGMRES-E(m,k)
1. Step 1 to step 7 of algorithm 2.
2. Decrease (increase) the value of each dimension d of vector x which in c previous iterations of the algorithm steadily decrease (increase), by $\beta * x'(d)$, where β is a real number that is called the update rate and $x'(d)$ is the amount of change in the value of vector x in dimension d in the before iteration.
3. Goto step 1.

By changing the step 8 of algorithm 3, the Improved SGMRES-DR(m,k) (ISGMRES-DR(m,k)) is obtained. The implementation of ISGMRES-DR(m,k) is outlined in Algorithm 6.

Algorithm 6: The template of ISGMRES-E(m,k)
1. Step 1 to step 7 of algorithm 3.
2. Decrease (increase) the value of each dimension d of vector x which in c previous iterations of the algorithm steadily decrease (increase), by $\beta * x'(d)$, where β is a real number that is called the update rate and $x'(d)$ is the amount of change in the value of vector x in dimension d in the before iteration.
3. Goto step 1.

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4. Experimental results

To investigate the performance of proposed methods, several experiments on a number of standard matrices of the University of Florida has been carried out. In all these experiments, the vector x_0 is initialized by zero, and computational error $\|b - A * x\| / \|b\|$ intended. In Table 1, SGMRES(m) and ISGMRES(m) are compared. According to Table 1, Figure1, and Figure 2, the performance of ISGMRES(m) is better than SGMRES(m).

Table 1: Comparison of the results of SGMRES(m) and ISGMRES(m)

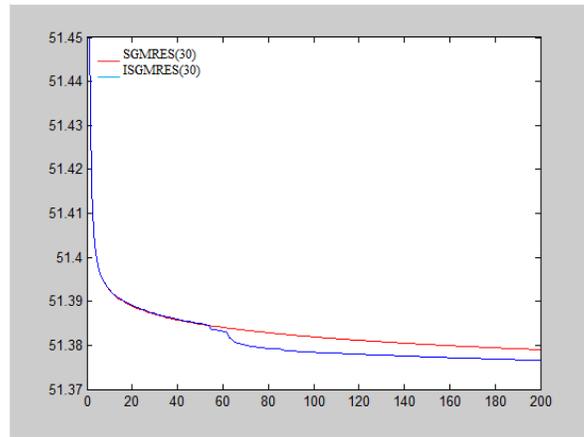| Problem	m	error	SGMRES	ISGMRES		
			iteration	c	β	iteration
Tols1090	20	0.2375	200	10	0.05	**144**
Tols1090	20	0.1847	500	10	0.007	**458**
Zenios	30	0.9585	200	5	0.00006	**84**
Zenios	30	0.9584	500	5	0.00007	**252**
Qh148	20	0.9898	200	5	0.005	**200**
Qh148	20	0.9898	500	5	0.9	**257**

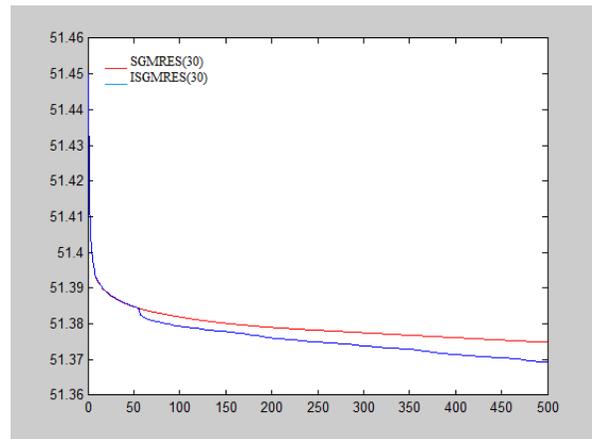Figure 1: Comparison of convergence of SGMRES(m) and ISGMRES(m) on Zenios matrix with up to 200 iteration

Figure 2: Comparison of convergence of SGMRES(m) and ISGMRES(m) on Zenios matrix with up to 500 iteration

In Table 2, SGMRES-E(m,k) and ISGMRES-E(m,k) is compared. According to Table 2,Figure3, and Figure 4, the performance of ISGMRES-E(m,k) is better than SGMRES-E(m,k).

Table 2: Comparison of the results of SGMRES-E(m,k) and ISGMRES-E(m,k)

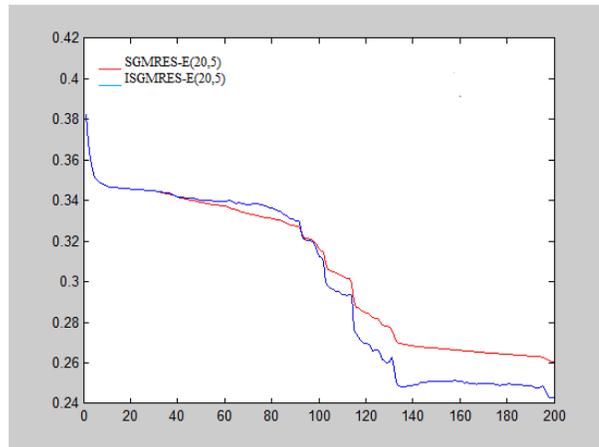| Problem	m	k	error	SGMRES-E	ISGMRES-E		
				iteration	c	β	iteration
Tols1090	20	5	0.2603	200	20	0.8	**128**
Tols1090	20	5	0.2063	500	20	0.4	**408**
Zenios	30	10	0.9574	200	5	0.00005	**182**
Zenios	30	10	0.9570	500	10	0.0005	**500**
Qh148	20	5	0.9758	200	1	0.006	**190**
Qh148	20	5	0.9750	500	15	0.1	**500**

Figure 3: Comparison of convergence of SGMRES-E(m,k) and ISGMRES-E(m,k) on Tols1090 matrix with up to 200 iteration
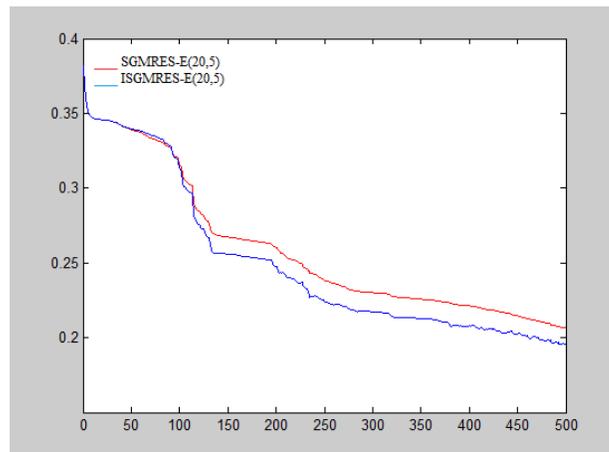

Figure 4: Comparison of convergence of SGMRES-E(m,k) and ISGMRES-E(m,k) on Tols1090 matrix with up to 500 iteration

In Table 3, SGMRES-DR(m,k) and ISGMRES-DR(m,k) is compared. According to Table 3,Figure5, and Figure 6, the performance of ISGMRES-DR(m,k) is better than SGMRES-DR(m,k).

Table 3: Comparison of the results of SGMRES-DR(m,k) and ISGMRES-DR(m,k)

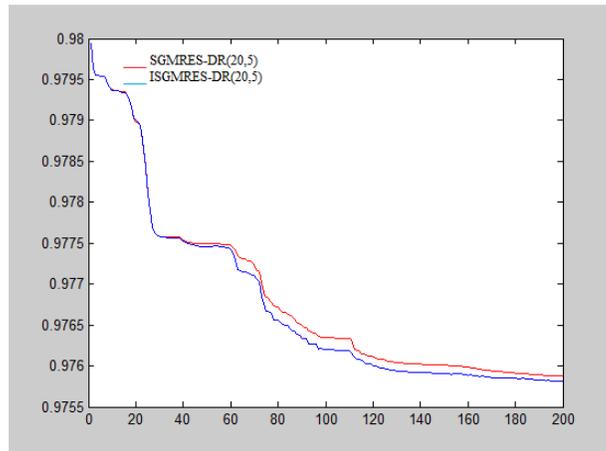| Problem	m	k	error	SGMRES-DR	ISGMRES-DR		
				iteration	c	β	iteration
Tols1090	20	5	0.3292	200	10	0.3	119
Tols1090	20	5	0.0327	500	5	0.005	355
Zenios	30	10	0.9576	200	10	0.00005	182
Zenios	30	10	0.9575	500	10	0.0005	500
Qh148	20	5	0.9758	200	5	0.04	164
Qh148	20	5	0.9757	500	5	0.02	453

Figure 5: Comparison of convergence of SGMRES-DR(m,k) and ISGMRES-DR(m,k) on Qh148 matrix
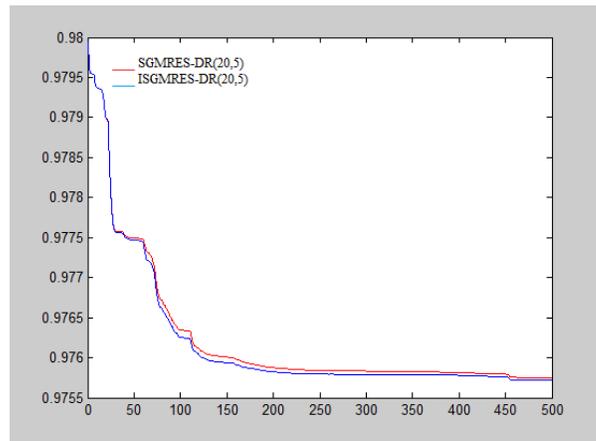with up to 200 iteration

Figure 6: Comparison of convergence of SGMRES-DR(m,k) and ISGMRES-DR(m,k) on Qh148 matrix
with up to 500 iteration

It should be noted that the computational complexity of each iteration of ISGMRES(m), ISGMRES-E(m), and ISGMRES-DR(m) is approximately equal to each iteration of SGMRES(m), ISGMRES-E(m), and ISGMRES-DR(m), respectively.

As it can be observed from the above tables and figures, the proposed methods compared with corresponding standard methods for finding solutions with the same error require a lesser number of iterations.

5. CONCLUSIONS

Based on the mathematical techniques, to improve the performance of SGMRES(m), SGMRES-E(m, k) and SGMRES-DR(m, k) methods, [5] - [8] - [7] - [13], that each of which have their own computational complexity and sometimes execution time and number of operations in the matrix vector multiplication greatly increase, a modified technique is proposed. Although, improving the

efficiency of these algorithms by using the concepts and techniques of artificial intelligence has received little attention, in this paper, an intelligent heuristic method that has very little computational complexity is used to improve the performance of procedures SGMRES(m), SGMRES-E(m, k) and SGMRES-DR(m, k).The numerical results obtained from implementation of the proposed method on several University of Florida standard matrixes confirm the efficiency of the proposed method.

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