# AN ANALYSIS OF THE FAT-TAILEDNESS OF THE CENTRALITY DISTRIBUTIONS OF REAL-WORLD NETWORKS

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***ABSTRACT***

*"Kurtosis" has long been considered an appropriate measure to quantify the extent of fat-tailedness of the degree distribution of a complex real-world network. However, the Kurtosis values for more than one real-world network have not been studied in conjunction with other statistical measures that also capture the extent of variation in node degree. Also, the Kurtosis values of the distributions of other commonly centrality metrics for real-world networks have not been analyzed. In this paper, we determine the Kurtosis values for a suite of 48 real-world networks along with measures such as SPR(K), Max(K)-Min(K), Max(K)-Avg(K), SD(K)/Avg(K), wherein SPR(K), Max(K), Min(K), Avg(K) and SD(K) represent the spectral radius ratio for node degree, maximum node degree, minimum node degree, average and standard deviation of node degree respectively. Contrary to the conceived notion in the literature, we observe that real-world networks whose degree distribution is Poisson in nature (characterized by lower values of SPR(K), Max(K)-Min(K), Max(K)-Avg(K), SD(K)/Avg(K)) could have Kurtosis values that are larger than that of real-world networks whose degree distribution is scale-free in nature (characterized by larger values of SPR(K), Max(K)-Min(K), Max(K)-Avg(K), SD(K)/Avg(K)). We also observe the Kurtosis values of the betweenness centrality distributions of the real-world networks to be more likely the largest among the Kurtosis values with respect to the commonly studied centrality metrics.*

***KEYWORDS***

*Fat-tailedness, Degree Distribution, Kurtosis, Real-World Networks, Centrality Metrics, Concordance*

## 1. INTRODUCTION

Complex network analysis is about analyzing complex real-world networks from a graph theoretic perspective [1]. Several measures from Statistics are also used to infer the distribution of the node-level metrics [2]. One such metric and distribution that is of interest in this paper is the degree centrality metric and the fat-tailedness of its distribution. The degree of a vertex is the number of neighbours for the vertex. A degree distribution is considered to be fat-tailed if the maximum degree of a vertex is much different from the minimum or the average degree of the vertex (correspondingly, the standard deviation of node degree is also comparable or even larger than that of the average node degree) [3]. Poisson degree distributions (characteristic of random networks [4]) are not fat-tailed; whereas, power-law degree distributions (characteristic of scale-free networks [5]) are fat-tailed. Real-world networks typically exhibit power-law degree distribution [3]; however, the extent of fat-tailedness of the distribution differs among the networks.

Until now, the Kurtosis measure has been perceived to be the most appropriate measure that could be used to quantify the extent of fat-tailedness of the degree distribution of the vertices in a real-world network [2]. But, there is no formal work that determined the Kurtosis of a suite of real-world networks of diverse degree distributions and analyzed whether the Kurtosis of a

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network with smaller variation in node degree (i.e., less fat-tailed) is more likely to be larger than the Kurtosis of a network with a relatively larger variation in node degree (i.e., more fat-tailed).
In this paper, we measure the Kurtosis of the degree distributions for a suite of 48 real-world networks in conjunction with several other relevant metrics that also capture the extent of variation in node degree. Let SPR($K$), Max($K$), Min($K$), Avg($K$) and SD($K$) represent the spectral radius ratio for node degree, maximum node degree, minimum node degree, average and standard deviation of node degree respectively. The metrics that are explored in this research along with Kurtosis for node degree are SPR($K$), Max($K$)-Min($K$), Max($K$)-Avg($K$) and SD($K$)/Avg($K$). The spectral radius ratio for node degree (SPR($K$)) [6] is defined as the ratio of the principal eigenvalue of the adjacency matrix of the network graph to that of the average node degree. According to literature [7], Min($K$) ≤ Avg($K$) ≤ Principal Eigenvalue($K$) ≤ Max($K$). The smaller the difference between Max($K$) and Min($K$) for a network, the lower the value for SPR($K$) = Principal Eigenvalue($K$) / Avg($K$). SPR($K$) values start from 1.0 and this is the value expected for a truly random network.

We seek to explore whether or not a real-world network A with larger Kurtosis for node degree than a real-world network B also incurs larger values for one of these above metrics that also capture the extent of variation in node degree. We measure the Kendall's concordance-based correlation coefficient [8] for Kurtosis with each of the above four metrics for the suite of 48 real-world networks. We say two networks A and B are concordant with respect to any two metrics (say, X and Y) if X(A) < X(B) and Y(A) < Y(B) or X(A) > X(B) and Y(A) > Y(B) or X(A) = X(B) and Y(A) = Y(B). Surprisingly, we observe Kendall's concordance-based correlation coefficient for Kurtosis with each of SPR($K$), Max($K$)-Min($K$), Max($K$)-Avg($K$) and SD($K$)/Avg($K$) to be low: 0.40, 0.26, 0.34 and 0.50 respectively; thus, seriously raising the question of using Kurtosis to compare the extent of fat-tailedness of the degree distribution of real-world networks when it has lower correlation with metrics that also capture the extent of variation in node degree.

There has been no analysis done on the Kurtosis of the distributions of the other commonly studied centrality metrics (such as eigenvector centrality, EVC; closeness centrality, CLC; betweenness centrality, BWC and the recently proposed local clustering coefficient complement-based degree centrality, LCC'DC). It is not clear which of these centrality metrics typically have the largest Kurtosis values. In the second half of the paper, we compute the Kurtosis values for the distributions of the other four commonly studied centrality metrics (as listed earlier: EVC, CLC, BWC and LCC'DC). We observe the Kurtosis for the BWC metric to be typically the largest among the centrality metrics, including degree centrality (DEG), for more than 75% of the real-world networks; on the other hand, the Kurtosis for the CLC metrics typically appears to be the smallest among the centrality metrics for slightly larger than 50% of the networks.

The rest of the paper is organized as follows: Section 2 illustrates the computation of Kurtosis for a real-world network with an example graph. Section 3 illustrates the computation of Kendall's concordance-based correlation coefficient (Kurtosis vs. Spectral radius ratio for node degree) for a subset of 8 real-world networks from the suite of 48 real-world networks studied in this research. Section 4 first provides a brief overview of the 48 real-world networks and then presents the values for Kurtosis and the other metrics (stated above) that capture the extent of variation in node degree. Section 4 also discusses the correlation for Kurtosis with each of these metrics. Section 5 presents an analysis of the Kurtosis values obtained for the commonly studied centrality metrics and identifies the centrality metric whose distribution is the fat-tailed for most of the real-world networks. Section 6 discusses related work and Section 7 concludes the paper. Throughout the paper, the terms 'node' and 'vertex', 'link' and 'edge', 'network' and 'graph' are used interchangeably. They mean the same.

## 2. KURTOSIS: FORMULATION AND ILLUSTRATION

Kurtosis has been traditionally used to quantify the extent of fat-tailedness of the distribution of a random variable. In the context of complex network analysis, Kurtosis has been used to quantify the extent of fat-tailedness of the degree distribution of the vertices in a real-world network. However, there is no formal work that has evaluated its appropriateness for comparing two real-world networks on the basis of the fat-tailedness of the degree distribution of the vertices in conjunction with other statistical and spectral metrics that also capture the extent of variation in node degree. In this section, we first present the formulation to compute the Kurtosis of the degree distribution of the vertices and then illustrate the computation with an example graph.

| | |
|---|---|
| 0 | 1 |
| 0 | 2 |
| 0 | 3 |
| 0 | 4 |
| 0 | 5 |
| 0 | 7 |
| 0 | 8 |
| 1 | 3 |
| 1 | 5 |
| 1 | 6 |
| 1 | 9 |
| 3 | 4 |
| 4 | 6 |
| 4 | 7 |
| 5 | 9 |
| 7 | 8 |

**Edge List**

0 → 1, 2, 3, 4, 5, 7, 8
1 → 0, 3, 5, 6, 9
2 → 0
3 → 0, 1, 4
4 → 0, 3, 6, 7
5 → 0, 1, 9
6 → 1, 4
7 → 0, 4, 8
8 → 0, 7
9 → 1, 5

| ID | Degree |
|---|---|
| 0 | 7 |
| 1 | 5 |
| 2 | 1 |
| 3 | 3 |
| 4 | 4 |
| 5 | 3 |
| 6 | 2 |
| 7 | 3 |
| 8 | 2 |
| 9 | 2 |

| Degree | #Nodes | P(K) |
|---|---|---|
| 1 | 1 | 1/10 = 0.1 |
| 2 | 3 | 3/10 = 0.3 |
| 3 | 3 | 3/10 = 0.3 |
| 4 | 1 | 1/10 = 0.1 |
| 5 | 1 | 1/10 = 0.1 |
| 7 | 1 | 1/10 = 0.1 |

Avg. Degree = $\sum_k k * P(k)$

Avg. Degree, Avg(K)
= (1)(0.1) + (2)(0.3) + (3)(0.3)
+ (4)(0.1) + (5)(0.1) + (7)(0.1)
= 3.2

| Degree (k) | P(k) | (k-Avg(K))$^2$ | (k-Avg(K))$^4$ | P(k)*(k-Avg(K))$^2$ | P(k)*(k-Avg(K))$^4$ |
|---|---|---|---|---|---|
| 1 | 0.1 | 4.84 | 23.43 | 0.484 | 2.343 |
| 2 | 0.3 | 1.44 | 2.07 | 0.432 | 0.621 |
| 3 | 0.3 | 0.04 | 0.0016 | 0.012 | 0.00048 |
| 4 | 0.1 | 0.64 | 0.4096 | 0.064 | 0.04096 |
| 5 | 0.1 | 3.24 | 10.498 | 0.324 | 1.0498 |
| 7 | 0.1 | 14.44 | 208.51 | 1.444 | 20.851 |

$$\textbf{SD(K)} = \sqrt{\sum_k P(k)*(k-Avg(K))^2} = \sqrt{2.76} = \underline{\mathbf{1.66}}$$

$$\textbf{Kurtosis(K)} = \frac{\sum_k P(k)*(k-Avg(K))^4}{SD^4} = \frac{24.91}{(1.66)^4} = \underline{\mathbf{3.27}}$$

Figure 1. Example Illustration to Compute the Average, Standard Deviation and Kurtosis of the Degree Distribution of the Vertices in a Graph

Let $K$ be the set of all degree values for the vertices in a graph. Let P($k$) indicate the probability of finding a vertex with degree $k$, where $k \in K$. The average, standard deviation and kurtosis for node degree are computed as follows:

$$\mathrm{Avg}(K) = \sum_{k \in K} P(k) * (k) \text{ .................................} \quad (1)$$

$$\mathrm{SD}(K) = \sqrt{\sum_{k \in K} P(k) * (k - Avg(K))^2} \text{ ............} \quad (2)$$

$$\mathrm{Kurtosis}(K) = \frac{\sum_{k \in K} P(k) * (k - Avg(K))^4}{SD(K)^4} \text{ ...........} \quad (3)$$

Figure 1 presents an illustration of the computation of the Avg($K$), SD($K$) and Kurtosis($K$) for node degree for an example undirected graph of 10 vertices (whose list of edges is given). We first compute the node degree (the number of neighbours for a vertex) and determine the number of vertices that are of a particular degree. The probability of finding a vertex with a certain degree is simply the fraction of the total number of vertices with the particular degree. Once we have the $k$ vs. P($k$) values for a graph, we can compute the above three statistical metrics using formulations (1), (2) and (3).

## 3. Kendall's Concordance-Based Correlation

The Kendall's concordance-based correlation measure could be used to evaluate the relative ranking of two networks with respect to any two network-level metrics; in our case, Kurtosis vs. any statistical or spectral metric. In this section, we illustrate the computation of Kendall's concordance-based correlation coefficient for a set of 8 real-world networks (taken from the suite of 48 real-world networks analyzed in Section 4) with respect to Kurtosis and Spectral radius ratio for node degree. Figure 2 illustrates the calculations. We count the number of concordat pairs of networks and the number of discordant pairs of networks and calculate Kendall's correlation coefficient as the ratio of the sum of the number of concordant pairs and discordant pairs to that of the difference of the number of concordant pairs and discordant pairs.

| Net. # | Net. Name | Kurtosis (K) | SPR(K) |
|---|---|---|---|
| 1 | ADJ | 15.41 | 1.73 |
| 8 | DON | 2.25 | 1.40 |
| 15 | FON | 5.89 | 1.01 |
| 23 | KCN | 6.30 | 1.47 |
| 25 | LMN | 8.89 | 1.82 |
| 33 | PBN | 4.35 | 1.42 |
| 36 | SJN | 6.34 | 1.29 |
| 44 | APN | 12.77 | 3.22 |

| Net. Pairs | Kurtosis (K) Values | SPR(K) Values | C/D |
|---|---|---|---|
| 1, 8 | 15.41, 2.25 | 1.73, 1.40 | C |
| 1, 15 | 15.41, 5.89 | 1.73, 1.01 | C |
| 1, 23 | 15.41, 6.30 | 1.73, 1.47 | C |
| 1, 25 | 15.41, 8.89 | 1.73, 1.82 | D |
| 1, 33 | 15.41, 4.35 | 1.73, 1.42 | C |
| 1, 36 | 15.41, 6.34 | 1.73, 1.29 | C |
| 1, 44 | 15.41, 12.77 | 1.73, 3.22 | D |
| 8, 15 | 2.25, 5.89 | 1.40, 1.01 | D |
| 8, 23 | 2.25, 6.30 | 1.40, 1.47 | C |
| 8, 25 | 2.25, 8.89 | 1.40, 1.82 | C |
| 8, 33 | 2.25, 4.35 | 1.40, 1.42 | C |
| 8, 36 | 2.25, 6.34 | 1.40, 1.29 | D |
| 8, 44 | 2.25, 12.77 | 1.40, 3.22 | C |

| Net. Pairs | Kurtosis Values | SPR(K) Values | C/D |
|---|---|---|---|
| 15, 23 | 5.89, 6.30 | 1.01, 1.47 | C |
| 15, 25 | 5.89, 8.89 | 1.01, 1.82 | C |
| 15, 33 | 5.89, 4.35 | 1.01, 1.42 | D |
| 15, 36 | 5.89, 6.34 | 1.01, 1.29 | C |
| 15, 44 | 5.89, 12.77 | 1.01, 3.22 | C |
| 23, 25 | 6.30, 8.89 | 1.47, 1.82 | C |
| 23, 33 | 6.30, 4.35 | 1.47, 1.42 | C |
| 23, 36 | 6.30, 6.34 | 1.47, 1.29 | D |
| 23, 44 | 6.30, 12.77 | 1.47, 3.22 | C |
| 25, 33 | 8.89, 4.35 | 1.82, 1.42 | C |
| 25, 36 | 8.89, 6.34 | 1.82, 1.29 | C |
| 25, 44 | 8.89, 12.77 | 1.82, 3.22 | C |
| 33, 36 | 4.35, 6.34 | 1.42, 1.29 | D |
| 33, 44 | 4.35, 12.77 | 1.42, 3.22 | C |
| 36, 44 | 6.34, 12.77 | 1.29, 3.22 | C |

#. Concordant Pairs = 21
#. Discordant Pairs = 7

Kendall's Concordance based Correlation Coefficient = $\frac{\text{\# Concordant Pairs} - \text{\# Discordant Pairs}}{\text{\# Concordant Pairs} + \text{\# Discordant Pairs}}$

Kendall's Concordance based Correlation Coefficient = $\frac{21 - 7}{21 + 7} = \frac{14}{28} = 0.50$

Figure 2. Example Illustration to Compute Kendall's Concordance-based Correlation Coefficient between Kurtosis and Spectral Radius Ratio for Node Degree for a Subset of the Real-World Networks

A pair of networks X and Y are said to be concordant with respect to Kurtosis($K$) and SPR($K$) if either one of the following is true:

(i) $\text{Kurtosis}_X(K) < \text{Kurtosis}_Y(K)$ and $\text{SPR}_X(K) < \text{SPR}_Y(K)$ or
(ii) $\text{Kurtosis}_X(K) > \text{Kurtosis}_Y(K)$ and $\text{SPR}_X(K) > \text{SPR}_Y(K)$ or
(iii) $\text{Kurtosis}_X(K) = \text{Kurtosis}_Y(K)$ and $\text{SPR}_X(K) = \text{SPR}_Y(K)$

A pair of networks X and Y are said to be discordant with respect to Kurtosis($K$) and SPR($K$) if either one of the following are true:

(i) $\text{Kurtosis}_X(K) > \text{Kurtosis}_Y(K)$ and $\text{SPR}_X(K) \leq \text{SPR}_Y(K)$ or
(ii) $\text{Kurtosis}_X(K) < \text{Kurtosis}_Y(K)$ and $\text{SPR}_X(K) \leq \text{SPR}_Y(K)$

For the set of 8 real-world networks considered in Figure 2 and their Kurtosis($K$) and SPR($K$) values, we observe 21 concordant pairs of networks and 7 discordant pairs of networks; this leads to Kendall's concordance-based correlation coefficient of (21-7) / (21+7) = 0.50.

## 4. Real-World Networks and their Correlation Analysis

In this section, we first introduce the 48 real-world networks analyzed in this paper. Table 1 lists the three character code acronym, name and the network type as well as the number of nodes and edges. The networks considered cover a broad range of categories (as listed below along with the number of networks in each category): Acquaintance network (12), Friendship network (9), Co-appearance network (6), Employment network (4), Citation network (3), Literature network (3), Collaboration network (2), Political network (2), Biological network (2), Game network (2), Geographical Network, Transportation network and Trade network (1 each). A brief description of each category of networks is as follows: An *acquaintance network* is a kind of social network in which the participant nodes slightly (not closely) know each other, as observed typically during an observation period. A *friendship network* is a kind of social network in which the participant nodes closely know each other and the relationship is not captured over an observation period. A *co-appearance network* is a network typically extracted from novels/books in such a way that two characters or words (modelled as nodes) are connected if they appear alongside each other. An *employment network* is a network in which the interaction/relationship between people is primarily due to their employment requirements and not due to any personal liking. A *citation network* is a network in which two papers (nodes) are connected if one paper cites the other paper as a reference. A *collaboration network* is a network of researchers/authors who are listed as co-authors in at least one publication. A *biological network* is a network that models the interactions between genes, proteins, animals of a species, etc. A *political network* is a network of entities (typically politicians) involved in politics. A *game network* is a network of teams or players playing for different teams and their associations. A *literature network* is a network of books/papers/terminologies/authors (other than collaboration, citation or co-authorship) involved in a particular area of literature. A *transportation network* is a network of entities (like airports and their flight connections) involved in public transportation. A *trade network* is a network of countries/people involved in a certain trade. The reader is referred to [9] for a more detailed description of the individual real-world networks.

Table 1. Real-World Networks used in the Correlation Analysis

| # | Net. | Net. Description | Ref. | Network Type | #nodes | #edges |
|---|---|---|---|---|---|---|
| 1 | ADJ | Word Adjacency Network | [10] | Co-appearance Net. | 112 | 425 |
| 2 | AKN | Anna Karnenina Network | [11] | Co-appearance Net. | 138 | 493 |
| 3 | JBN | Jazz Band Network | [12] | Employment Net. | 198 | 2742 |
| 4 | CEN | C. Elegans Neural | [13] | Biological Net. | 297 | 2148 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | Network | | | | |
| 5 | CLN | Centrality Literature Net. | [14] | Citation Net. | 129 | 613 |
| 6 | CGD | Citation Graph Drawing Net | [15] | Citation Net. | 311 | 640 |
| 7 | CFN | Copperfield Network | [11] | Co-appearance Net. | 87 | 406 |
| 8 | DON | Dolphin Network | [16] | Acquaintance Net. | 62 | 159 |
| 9 | DRN | Drug Network | [17] | Acquaintance Net. | 298 | 337 |
| 10 | DLN | Dutch Literature 1976 Net. | [18] | Literature Net. | 35 | 80 |
| 11 | ERD | Erdos Collaboration Net. | [19] | Collaboration Net. | 472 | 1314 |
| 12 | EUR | Euro Road Network | [51] | Geographical Net. | 1174 | 1417 |
| 13 | FMH | Faux Mesa High School Net | [20] | Friendship Net. | 205 | 404 |
| 14 | FHT | Friendship in Hi-Tech Firm | [21] | Friendship Net. | 36 | 147 |
| 15 | FTC | Flying Teams Cade Net. | [22] | Employment Net. | 48 | 170 |
| 16 | FON | US Football Network | [23] | Game Net. | 115 | 613 |
| 17 | CDF | College Dorm Fraternity Net | [24] | Acquaintance Net. | 58 | 967 |
| 18 | GD96 | Graph Drawing 1996 Net | [19] | Citation Net. | 180 | 229 |
| 19 | MUN | Marvel Universe Network | [25] | Co-appearance Net. | 165 | 300 |
| 20 | GLN | Graph Glossary Network | [19] | Literature Net. | 72 | 236 |
| 21 | HTN | Hypertext 2009 Network | [26] | Acquaintance Net. | 113 | 2163 |
| 22 | HCN | Huckleberry Coappear. Net. | [11] | Co-appearance Net. | 74 | 301 |
| 23 | ISP | Infectious Socio-Patterns Net | [26] | Acquaintance Net. | 309 | 1924 |
| 24 | KCN | Karate Club Network | [27] | Acquaintance Net. | 34 | 78 |
| 25 | KFP | Korea Family Planning Net. | [28] | Acquaintance Net. | 39 | 84 |
| 26 | LMN | Les Miserables Network | [11] | Co-appearance Net. | 77 | 254 |
| 27 | MDN | Macaque Dominance Net. | [29] | Biological Net. | 62 | 1167 |
| 28 | MTB | Madrid Train Bombing Net. | [30] | Acquaintance Net. | 70 | 486 |
| 29 | MCE | Manufact. Comp. Empl. Net. | [31] | Employment Net. | 77 | 2326 |
| 30 | MSJ | Soc. Net. Journal Co-authors | [32] | Co-author Net. | 475 | 625 |
| 31 | AFB | Author Facebook Network | - | Friendship Net. | 187 | 939 |
| 32 | MPN | Mexican Political Elite Net. | [33] | Political Net. | 35 | 117 |
| 33 | MMN | ModMath Network | [19] | Friendship Net. | 38 | 61 |

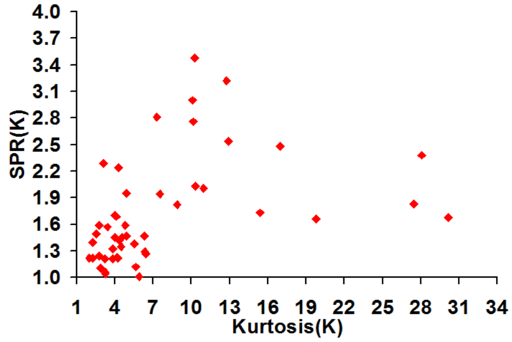
| | | | | | | |
|---|---|---|---|---|---|---|
| 34 | PBN | US Politics Books Network | [34] | Literature Net. | 105 | 441 |
| 35 | PSN | Primary School Contact Net. | [35] | Acquaintance Net. | 238 | 5539 |
| 36 | PFN | Prison Friendship Network | [36] | Friendship Net. | 67 | 182 |
| 37 | SJN | San Juan Sur Family Net. | [37] | Acquaintance Net. | 75 | 155 |
| 38 | SDI | Scotland Corp. Interlock Net | [38] | Employment Net. | 244 | 358 |
| 39 | SPR | Senator Press Release Net. | [39] | Political Net. | 92 | 477 |
| 40 | SWC | Soccer World Cup 1998 Net | [19] | Game Net. | 35 | 118 |
| 41 | SSM | Sawmill Strike Comm. Net. | [40] | Acquaintance Net. | 24 | 38 |
| 42 | TEN | Taro Exchange Network | [41] | Acquaintance Net. | 22 | 39 |
| 43 | TWF | Teenage Female Friend Net. | [42] | Friendship Net. | 50 | 122 |
| 44 | UKF | UK Faculty Friendship Net. | [43] | Friendship Net. | 81 | 577 |
| 45 | APN | US Airports 1997 Network | [19] | Transportation Net. | 332 | 2126 |
| 46 | RHF | Residence Hall Friend Net. | [45] | Friendship Net. | 43 | 336 |
| 47 | WSB | Windsurfers Beach Network | [46] | Friendship Net. | 80 | 875 |
| 48 | WTN | World Trade Metal Network | [47] | Trade Net. | 112 | 425 |

Table 2 lists the values for SPR(*K*), Avg(*K*), SD(*K*), Min(*K*), Max(*K*) and Kurtosis(*K*) obtained for these 48 real-world networks. Figure 3 plots the distribution of Kurtosis(*K*) vs. each of the following: SPR(*K*), SD(*K*)/Avg(*K*), Max(*K*) - Min(*K*) and Max(*K*) - Avg(*K*). We also mention the values for Kendall's correlation coefficient obtained for Kurtosis(*K*) vs. each of these metrics. We observe all the four correlation coefficient values to be less than or equal to 0.50; the largest being 0.50 for Kurtosis vs. SD(*K*)/Avg(*K*) ratio and the lowest being 0.26 for Kurtosis(*K*) vs. Max(*K*) - Min(*K*), an appreciable measure of the extent of variation in node degree and fat-tailedness nature of the degree distribution.

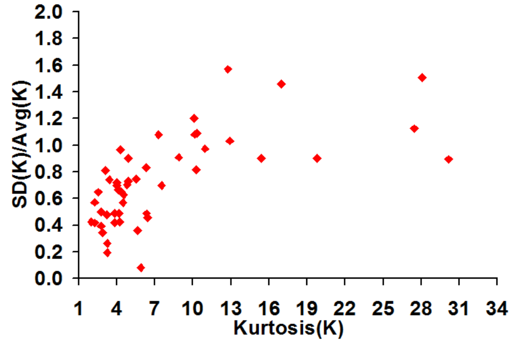
Table 2. SPR, Avg, SD, Min, Max and Kurtosis Values for the Degree Distribution of the Real-World Networks

| # | Net. | SPR(*K*) | Avg(*K*) | SD(*K*) | Min(*K*) | Max(*K*) | Kurtosis (*K*) |
|---|---|---|---|---|---|---|---|
| 1 | ADJ | 1.73 | 7.59 | 6.85 | 1 | 49 | 15.41 |
| 2 | AKN | 2.48 | 7.14 | 10.43 | 1 | 71 | 16.97 |
| 3 | JBN | 1.45 | 27.70 | 17.41 | 1 | 100 | 4.54 |
| 4 | CEN | 1.68 | 14.46 | 12.94 | 1 | 134 | 30.18 |
| 5 | CLN | 2.03 | 9.50 | 10.35 | 0 | 66 | 10.30 |
| 6 | CGD | 2.24 | 4.12 | 3.98 | 0 | 20 | 4.27 |

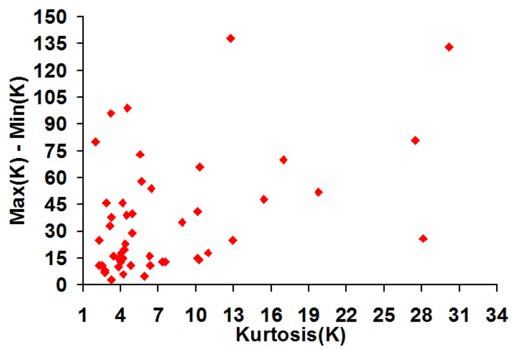
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 7 | CFN | 1.83 | 9.33 | 10.49 | 1 | 82 | 27.46 |
| 8 | DON | 1.40 | 5.13 | 2.93 | 1 | 12 | 2.25 |
| 9 | DRN | 2.76 | 1.91 | 2.06 | 0 | 15 | 10.12 |
| 10 | DLN | 1.49 | 4.57 | 2.96 | 1 | 12 | 2.52 |
| 11 | ERD | 3.00 | 5.57 | 6.69 | 0 | 41 | 10.11 |
| 12 | EUR | 1.66 | 2.41 | 1.19 | 1 | 10 | 7.03 |
| 13 | FMH | 2.81 | 1.97 | 2.12 | 0 | 13 | 7.29 |
| 14 | FHT | 1.57 | 5.06 | 3.74 | 0 | 16 | 3.41 |
| 15 | FTC | 1.21 | 7.08 | 2.97 | 1 | 16 | 3.82 |
| 16 | FON | 1.01 | 10.66 | 0.88 | 7 | 12 | 5.89 |
| 17 | CDF | 1.11 | 33.34 | 11.43 | 6 | 52 | 2.87 |
| 18 | GD96 | 2.38 | 2.53 | 3.82 | 1 | 27 | 28.07 |
| 19 | MUN | 2.54 | 3.64 | 3.76 | 1 | 26 | 12.92 |
| 20 | GLN | 2.01 | 3.28 | 3.19 | 0 | 18 | 10.96 |
| 21 | HTN | 1.21 | 38.28 | 18.30 | 1 | 97 | 3.21 |
| 22 | HCN | 1.66 | 8.14 | 7.34 | 1 | 53 | 19.77 |
| 23 | ISP | 1.69 | 12.45 | 8.33 | 1 | 47 | 4.14 |
| 24 | KCN | 1.47 | 4.59 | 3.82 | 1 | 17 | 6.30 |
| 25 | KFP | 1.70 | 4.31 | 3.11 | 0 | 13 | 3.99 |
| 26 | LMN | 1.82 | 6.60 | 6.00 | 1 | 36 | 8.89 |
| 27 | MDN | 1.04 | 37.65 | 7.40 | 17 | 55 | 3.24 |
| 28 | MTB | 1.95 | 6.94 | 6.27 | 0 | 29 | 4.91 |
| 29 | MCE | 1.12 | 34.91 | 12.53 | 18 | 76 | 5.64 |
| 30 | MSJ | 3.48 | 2.63 | 2.15 | 1 | 15 | 10.25 |
| 31 | AFB | 2.29 | 10.04 | 8.16 | 0 | 33 | 3.11 |
| 32 | MPN | 1.23 | 6.69 | 3.27 | 2 | 17 | 4.18 |
| 33 | MMN | 1.59 | 3.21 | 2.26 | 0 | 11 | 4.81 |
| 34 | PBN | 1.42 | 8.40 | 5.45 | 2 | 25 | 4.35 |
| 35 | PSN | 1.22 | 46.55 | 19.85 | 8 | 88 | 2.00 |
| 36 | PFN | 1.32 | 4.24 | 2.07 | 1 | 11 | 3.83 |
| 37 | SJN | 1.29 | 4.13 | 2.02 | 1 | 12 | 6.34 |
| 38 | SDI | 1.94 | 2.93 | 2.04 | 0 | 13 | 7.53 |
| 39 | SPR | 1.57 | 10.37 | 7.55 | 1 | 41 | 4.91 |
| 40 | SWC | 1.45 | 6.74 | 4.71 | 1 | 19 | 4.02 |
| 41 | SSM | 1.22 | 3.17 | 1.34 | 1 | 7 | 4.20 |
| 42 | TEN | 1.06 | 3.55 | 0.94 | 3 | 6 | 3.24 |
| 43 | TWF | 1.49 | 3.08 | 1.55 | 0 | 7 | 2.75 |
| 44 | UKF | 1.35 | 14.25 | 8.11 | 2 | 41 | 4.48 |
| 45 | APN | 3.22 | 12.81 | 20.10 | 1 | 139 | 12.77 |
| 46 | RHF | 1.27 | 16.95 | 7.76 | 2 | 56 | 6.42 |
| 47 | WSB | 1.16 | 15.63 | 6.53 | 6 | 31 | 2.26 |
| 48 | WTN | 1.38 | 21.88 | 16.33 | 4 | 77 | 5.54 |



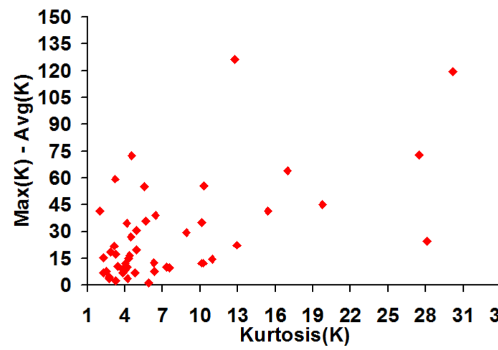

Kurtosis($K$) vs. SPR($K$)
Kendall's Correlation Coefficient = 0.40




Kurtosis($K$) vs. SD($K$)/Avg($K$)
Kendall's Correlation Coefficient = 0.50




Kurtosis($K$) vs. Max($K$) - Min($K$)
Kendall's Correlation Coefficient = 0.26




Kurtosis($K$) vs. Max($K$) - Avg($K$)
Kendall's Correlation Coefficient = 0.35

Figure 3. Distribution of the Kurtosis($K$) Values vs. {SPR($K$), SD($K$)/Avg($K$), Max($K$)-Min($K$) and Max($K$)-Avg($K$)} Values and the Kendall's Correlation Coefficient for the 48 Real-World Networks

## 5. KURTOSIS ANALYSIS FOR CENTRALITY METRICS

In this section, we present a comparison of the Kurtosis values for five centrality metrics (including degree centrality) that are widely applicable for complex network analysis. The centrality metrics considered (in addition to degree centrality, represented as DEG in this section) are: eigenvector centrality (EVC), closeness centrality (CLC), betweenness centrality (BWC) and the local clustering coefficient complement-based degree centrality (LCC'DC). The EVC [52] of a vertex is a measure of the degree of the vertex as well as the degree of its neighbors. The CLC [53] of a vertex is a measure of the sum of the distances (typically the number of hops on the shortest path) of the vertex to the rest of the vertices in the network. The BWC [54] of a vertex is a measure of the fraction of the shortest paths that the vertex lies on between any two vertices in the network. While DEG and EVC are degree-based centrality metrics, CLC and BWC are shortest path-based metrics. The LCC'DC metric [55] is a computationally-light alternative to the computationally-heavy BWC metric. The LCC'DC (a hybrid of both the degree and shortest path-based metrics) of a vertex is computed as the product of the degree centrality of the vertex and the probability that any two neighbors of the vertex go through the vertex for shortest path communication (the latter is computed as the ratio of the number of pairs of neighbors of the vertex that are not directly connected to each other to that of the maximum number of pairs of neighbors of the vertex that could be directly connected to each other). Table 3 presents the Kurtosis values for the distributions of the five centrality metrics (DEG, EVC, CLC, BWC and LCC'DC) obtained for the 48 real-world networks.

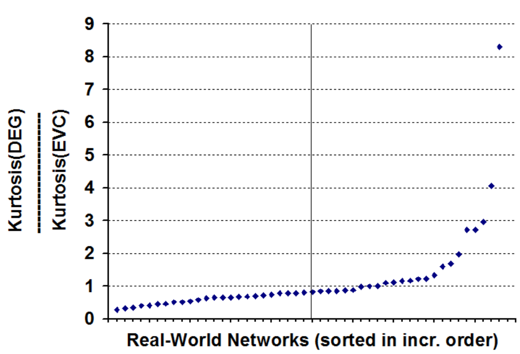
Table 3. Kurtosis Values of the Centrality Metrics for Real-World Networks

| # | Net. | DEG | EVC | CLC | BWC | LCC'DC | Max. | Min. | Max.-Min. |
|---|---|---|---|---|---|---|---|---|---|
| 1 | ADJ | 15.41 | 10.18 | 3.69 | 35.47 | 16.52 | BWC | CLC | 31.77 |
| 2 | AKN | 16.97 | 7.95 | 5.63 | 41.22 | 25.60 | BWC | CLC | 35.59 |
| 3 | JBN | 4.54 | 3.09 | 4.53 | 60.71 | 9.84 | BWC | EVC | 57.63 |
| 4 | CEN | 30.18 | 9.74 | 3.96 | 216.69 | 46.93 | BWC | CLC | 212.73 |
| 5 | CLN | 10.30 | 5.28 | 9.82 | 34.01 | 18.09 | BWC | EVC | 28.72 |
| 6 | CGD | 4.27 | 8.47 | 3.27 | 15.54 | 5.05 | BWC | CLC | 12.28 |
| 7 | CFN | 27.46 | 7.84 | 32.21 | 81.72 | 48.70 | BWC | EVC | 73.88 |
| 8 | DON | 2.25 | 2.60 | 2.46 | 7.52 | 2.49 | BWC | DEG | 5.27 |
| 9 | DRN | 10.12 | 27.53 | 1.38 | 24.77 | 8.89 | BWC | CLC | 26.15 |
| 10 | DLN | 2.52 | 2.16 | 2.31 | 2.81 | 2.46 | BWC | EVC | 0.65 |
| 11 | ERD | 10.11 | 12.41 | 9.08 | 22.79 | 12.09 | BWC | CLC | 13.72 |
| 12 | EUR | 7.03 | 58.35 | 6.82 | 21.25 | 5.29 | EVC | LCC'DC | 53.06 |
| 13 | FMH | 7.29 | 21.62 | 1.12 | 16.35 | 7.32 | EVC | CLC | 20.50 |
| 14 | FHT | 3.41 | 2.79 | 10.09 | 3.33 | 2.53 | DEG | LCC'DC | 7.56 |
| 15 | FTC | 3.82 | 2.62 | 3.04 | 4.08 | 3.12 | BWC | EVC | 1.46 |
| 16 | FON | 5.89 | 2.41 | 3.06 | 3.88 | 3.49 | DEG | EVC | 3.48 |
| 17 | CDF | 2.87 | 3.20 | 2.45 | 3.41 | 4.41 | LCC'DC | CLC | 1.97 |
| 18 | GD96 | 28.07 | 16.21 | 3.34 | 26.77 | 25.44 | DEG | CLC | 24.73 |
| 19 | MUN | 12.92 | 12.83 | 1.63 | 54.50 | 14.32 | BWC | CLC | 52.87 |
| 20 | GLN | 10.96 | 7.23 | 4.20 | 15.22 | 8.79 | BWC | CLC | 11.02 |
| 21 | HTN | 3.21 | 2.63 | 5.04 | 29.59 | 10.49 | BWC | EVC | 26.96 |
| 22 | HCN | 19.77 | 8.02 | 12.87 | 60.78 | 24.91 | BWC | EVC | 52.77 |
| 23 | ISP | 4.14 | 4.84 | 2.78 | 31.93 | 5.20 | BWC | CLC | 29.14 |
| 24 | KCN | 6.30 | 3.24 | 2.20 | 11.31 | 6.35 | BWC | CLC | 9.11 |
| 25 | KFP | 3.99 | 2.77 | 7.86 | 2.95 | 3.24 | CLC | EVC | 5.09 |
| 26 | LMN | 8.89 | 3.04 | 4.51 | 43.90 | 13.36 | BWC | EVC | 40.86 |
| 27 | MDN | 3.24 | 3.21 | 3.23 | 3.74 | 3.14 | BWC | LCC'DC | 0.60 |
| 28 | MTB | 4.91 | 3.20 | 9.76 | 8.06 | 3.19 | CLC | LCC'DC | 6.57 |
| 29 | MCE | 5.64 | 3.03 | 9.31 | 14.61 | 4.27 | BWC | EVC | 11.58 |
| 30 | MSJ | 10.25 | 41.63 | 2.27 | 45.57 | 6.35 | BWC | CLC | 43.30 |
| 31 | AFB | 3.11 | 5.24 | 1.76 | 44.65 | 3.60 | BWC | CLC | 42.89 |
| 32 | MPN | 4.18 | 3.26 | 3.52 | 9.38 | 4.91 | BWC | EVC | 6.12 |
| 33 | MMN | 4.81 | 4.24 | 3.02 | 7.15 | 3.80 | BWC | CLC | 4.14 |
| 34 | PBN | 4.35 | 5.28 | 2.45 | 5.31 | 3.83 | BWC | CLC | 2.86 |
| 35 | PSN | 2.00 | 2.01 | 2.21 | 4.12 | 2.37 | BWC | DEG | 2.12 |
| 36 | PFN | 3.83 | 4.22 | 2.85 | 6.59 | 2.86 | BWC | CLC | 3.75 |
| 37 | SJN | 6.34 | 8.49 | 3.63 | 14.55 | 4.59 | BWC | CLC | 10.92 |
| 38 | SDI | 7.53 | 20.52 | 7.16 | 16.65 | 6.78 | BWC | LCC'DC | 13.73 |
| 39 | SPR | 4.91 | 3.68 | 2.86 | 14.17 | 6.97 | BWC | CLC | 11.31 |
| 40 | SWC | 4.02 | 2.87 | 3.00 | 5.72 | 2.69 | BWC | LCC'DC | 3.03 |
| 41 | SSM | 4.20 | 3.55 | 3.62 | 7.76 | 2.57 | BWC | LCC'DC | 5.19 |
| 42 | TEN | 3.24 | 2.55 | 2.17 | 2.88 | 1.96 | DEG | LCC'DC | 1.28 |

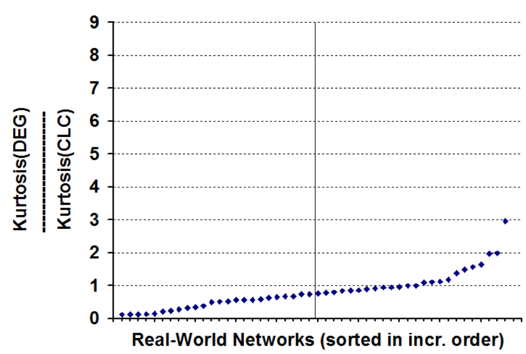
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 43 | TWF | 2.75 | 4.38 | 3.81 | 3.23 | 2.02 | EVC | LCC'DC | 2.37 |
| 44 | UKF | 4.48 | 2.81 | 3.52 | 17.34 | 6.93 | BWC | EVC | 14.52 |
| 45 | APN | 12.77 | 5.92 | 3.68 | 47.41 | 33.46 | BWC | CLC | 43.73 |
| 46 | RHF | 6.42 | 5.46 | 3.28 | 21.35 | 5.77 | BWC | CLC | 18.07 |
| 47 | WSB | 2.26 | 1.98 | 2.55 | 7.12 | 4.42 | BWC | EVC | 5.15 |
| 48 | WTN | 5.54 | 4.34 | 8.30 | 14.80 | 7.76 | BWC | EVC | 10.46 |

From Table 3, it is evident that the Kurtosis values for the BWC metric are the largest among the five centrality metrics for 38 of the 48 (i.e., more than 75%) of the real-world networks. The CLC metric incurs the smallest of the Kurtosis values for 23 of the 48 (i.e., about 48%) real-world networks. While one of the four centrality metrics (DEG, EVC, CLC and LCC'DC) exhibited the smallest of the Kurtosis values for at least one real-world network, the BWC metric did not incur the smallest of the Kurtosis values for any of the real-world networks. Hence, the distribution of the BWC metric for real-world networks could be confidently considered the most fat-tailed of the centrality metrics.

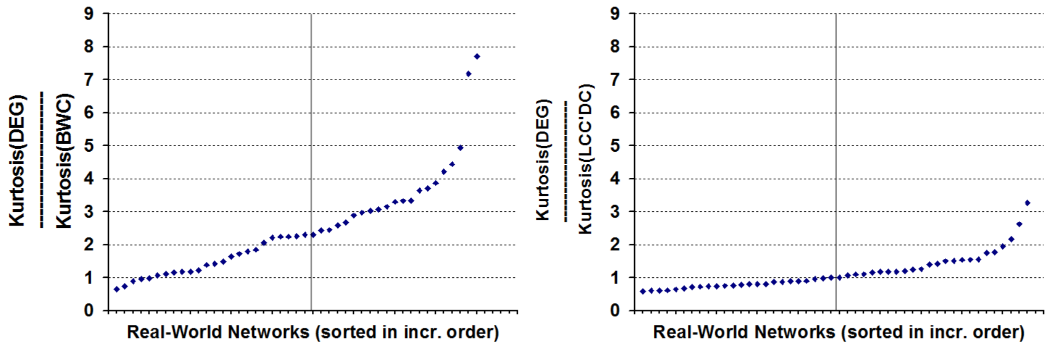
In Figure 4, we show the distribution of the Kurtosis(X)/Kurtosis(DEG) where X ∈ {EVC, CLC, BWC and LCC'DC} for the real-world networks. We observe the Kurtosis(BWC)/Kurtosis(DEG) ratios to be above 1 for 43 of the 48 (close to 90%) real-world networks, with a median of 2.30. On the other hand, the Kurtosis(EVC)/Kurtosis(DEG) and Kurtosis(CLC)/Kurtosis(DEG) ratios are below 1 for 32 and 37 of the 48 real-world networks (with medians of 0.82 and 0.75) respectively. In earlier works [55-56], the BWC and DEG metrics have been observed to exhibit a very strong correlation for several real-world networks. However, the above observations regarding the Kurtosis(DEG) and Kurtosis(BWC) values indicate that real-world networks are expected to be relatively more fat-tailed with respect to BWC compared to that of DEG. On the other hand, the lower values (lower than 1.0) for the Kurtosis(CLC)/Kurtosis(DEG) and Kurtosis(EVC)/Kurtosis(DEG) ratios for at least 2/3rds of the real-world networks indicate that real-world networks are likely to be less fat-tailed with respect to the EVC and CLC metrics vis-a-vis DEG. The median of the Kurtosis(DEG)/Kurtosis(LCC'DC) ratios is 1.00, indicating that the local clustering coefficient component of the LCC'DC formulation does not contribute to the fat-tailedness of the LCC'DC distribution. These are interesting observations that have been hitherto not reported in the literature.



Kurtosis(DEG) vs. Kurtosis(EVC)



Kurtosis(DEG) vs. Kurtosis(CLC)




Kurtosis(DEG) vs. Kurtosis(BWC) Kurtosis(DEG) vs. Kurtosis(LCC'DC)

Figure 4. Comparison of the Kurtosis Values for the Centrality Metrics

## 6. RELATED WORK

In the context of complex network analysis, Kurtosis has been typically used to capture the extent of fat-tailedness of degree distribution of the vertices and make an initial educated guess on the type (i.e., Poisson random networks or Power-law scale-free networks) of degree distribution for an underlying network graph [3]. A real-world network with Kurtosis for the degree distribution greater than 3 is typically considered to be fat-tailed [48]. Kurtosis has also been used to analyze the possibility of an existence of outlier(s) in a data set [49]. In the context of complex network analysis, a larger Kurtosis for the degree distribution of a network could imply that the network has one or more nodes with degree(s) that is extremely different from the rest of the nodes in the network [3]. But, the existence of few such outlier nodes is not sufficient to classify a network as a fat-tailed network. We would need the degree distribution to exhibit non-zero probability values for degree values spanning a broader range and exhibit a decreasing trend as the degree values approach the extreme value.

Instead of Kurtosis, several other approaches have also been attempted in the literature to capture the extent of variation in node degree (inclusive of fat-tailedness). For example, graph traversal algorithms like Breadth First Search (BFS) [50] have been used in the literature to analyze the fat-tailed nature of real-world networks. The BFS algorithm could be used to determine the diameter of a network. The idea proposed in [50] is to calculate the diameter ($D_0$) of the unperturbed network (with all nodes in the network) and calculate the diameter ($D_i$) of the network due to the removal of node $i$. The $\Delta_i = D_i - D_0/D_0$ value for each node is then calculated. A distribution of probability($\Delta_i$) vs. the $\Delta_i$ values (for $\Delta_i > 0$) is plotted and if it appears to mimic a power-law distribution, then the network is considered to be fat-free.

## 7. CONCLUSIONS

The high-level contribution of this paper is to illustrate that Kurtosis measure may not be appropriate to compare any two real-world networks with respect to the extent of fat-tailedness. The Kurtosis of a network with a lower variation in node degree (less fat-tailed) could be larger than the Kurtosis of a network with a relatively larger variation in node degree (relatively more fat-tailed). We measure the Kendall's concordance-based correlation coefficient for Kurtosis with four different statistical/spectral measures that effectively capture the variation in node degree. We observe the correlation coefficients to be no more than 0.50. From the analysis done in Section 5, we could confidently conclude that the distribution of the BWC metric is more likely to be most fat-tailed among all the centrality distributions for real-world networks. We could also conclude that the distributions of the EVC and CLC metrics are more likely to be relatively less fat-tailed compared to the distribution of the degree centrality metric.

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