SOLVING BIPOLAR MAX-T_P EQUATION CONSTRAINED MULTI-OBJECTIVE OPTIMIZATION PROBLEMS

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ABSTRACT

This work considers the multi-objective optimization problem constrained by a system of bipolar fuzzy relational equations with max-product composition. An integer optimization based technique for order of preference by similarity to the ideal solution is proposed for solving such a problem. Some critical features associated with the feasible domain and optimal solutions of the bipolar max-T_p equation constrained optimization problem are studied. An illustrative example verifying the idea of this paper is included. This is the first attempt to study the bipolar max-T equation constrained multi-objective optimization problems from an integer programming viewpoint.

KEYWORDS

Fuzzy relational equations, Bipolar constraints, Fuzzy optimization

1. INTRODUCTION

Fuzzy relational equations have played an important role in many applications of fuzzy sets and systems [10], [18]. Sanchez was the first to study fuzzy relational equations in terms of max-min composition for simulating cause and effect connections in medical diagnosis problems [19]. Since then, fuzzy relational equations based on various compositions, e.g., max-min, max-product, and max-Łukasiewicz t-norm compositions have been investigated and applied extensively[15], [16], [17]. The resolution of a system of fuzzy relational equations with max-T composition is to determine the unknown vector x for a given coefficient matrix A and a right hand side vector b such that $A \circ x = b$; where "$\circ$" stands for the specific max-T composition with T being a continuous triangular norm. The set of all solutions, when it is non-empty, is a finitely generated root system which can be fully determined by a unique maximum solution and a finite number of minimal solutions [10]. For a finite system of fuzzy relational equations with max-T composition, its consistency can be verified by constructing and checking a potential maximum solution. However, the detection of all minimal solutions is closely related to the set covering problem and remains a challenging problem. Overviews of fuzzy relational equations and their applications can be found in [10] and [18].

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The system of bipolar fuzzy relational equations with max-T composition has been considered as a generalization of the system of fuzzy relational equations, which can be expressed in the matrix form as

$$A^{+} \circ \mathrm{x} \vee A^{-} \circ \bar{\mathrm{x}} = \mathrm{b}, \tag{1}$$

where $\mathrm{x} = (x_1, x_2, \cdots, x_n)^T \in [0,1]^n$, $\bar{\mathrm{x}}$ denotes the logical negation of x, i.e., $\bar{\mathrm{x}} = (1-x_1, 1-x_2, \cdots, 1-x_n)^T$, $A^{+} = (a_{ij}^{+})_{m\times n} \in [0,1]^{m\times n}$, $A^{-} = (a_{ij}^{-})_{m\times n} \in [0,1]^{m\times n}$, $\mathrm{b} = (b_1, b_2, \cdots, b_m)^T \in [0,1]^m$, and " $\circ$ " stands for the max-$\mathcal{T}$ composition with $\mathcal{T}$ being a continuous triangular norm. The most frequently used triangular norm in applications of fuzzy relational equations is the minimum operator $\mathcal{T}_M$, i.e., $\mathcal{T}_M(x,y) = \min(x,y)$. Another two important triangular norms are the product operator $\mathcal{T}_P(x,y) = x \cdot y$ and the Łukasiewicz t-norm $\mathcal{T}_L(x,y) = \max(x+y-1, 0)$. It is clear that $A^{+} \circ \mathrm{x} \vee A^{-} \circ \bar{\mathrm{x}} = \mathrm{b}$ would degenerate into $A^{-} \circ \bar{\mathrm{x}} = \mathrm{b}$ or $A^{+} \circ \mathrm{x} = \mathrm{b}$ if either A^{+} or A^{-} is the zero matrix, respectively. Therefore, a system of max-$\mathcal{T}$ bipolar equations can be viewed as a generalization of fuzzy relational equations, containing the decision variables and their logical negations simultaneously.

The system of bipolar max-T_M equations and the associated linear optimization problem with a potential application of product public awareness in revenue management were first introduced by Freson et al. [2]. It was shown that the solution set of a system of bipolar max-T_M equations, whenever nonempty, can be characterized by a finite set of maximal and minimal solution pairs. However, as indicated by Li and Jin [12], determining the consistency of a system of bipolar max- T_M equations is NP-complete. Consequently, solving the bipolar max- T_M equation constrained linear optimization problem is inevitably NP-hard. Recently Li and Liu [11] showed that the problem of minimizing an linear objective function subject to a system of bipolar max-T_L equations can be reduced to a 0-1 integer programming problem in polynomial time.

Motivated by the recent research, this work considers the bipolar max-T_p equation constrained multi-objective optimization problem which can be expressed as

$$\begin{aligned} \text{Max/Min} \quad & F(\mathrm{x}) = [f_1(\mathrm{x}), \cdots, f_k(\mathrm{x}), \cdots, f_K(\mathrm{x})] \\ \text{s.t.} \quad & A^{+} \circ \mathrm{x} \vee A^{-} \circ \bar{\mathrm{x}} = \mathrm{b}, \end{aligned} \tag{2}$$

where $f_k : [0,1]^n \rightarrow \mathbf{R}, k = 1, 2, \cdots, K$, is a real-valued function, and " $\circ$ " stands for the max-product composition.

Multi-objective decision making (MODM) techniques have attracted a great deal of interest due to their adaptability to real-life decision making problems. Wang [21] firstly explored the fuzzy multi-objective linear programming subject to max-t norm composition fuzzy relation equations for medical applications. Guu et al. [3] proposed a two-phase method to solve a multiple objective optimization problem under a max-Archimedean t-norm fuzzy relational equation constraint. Loetamonphong et al. [14] provided a genetic algorithm to find the Pareto optimal solutions for a nonlinear multi-objective optimization problem with fuzzy relation equation constraints. Khorram and Zarei [9] considered a multiple objective optimization model subject to a system of fuzzy relation equations with max-average composition. It is well known that many decision making problems have multiple objectives which cannot be optimized simultaneously due to the inherent incommensurability and conflict among these objectives. Thus, making a trade off between these objectives becomes a major subject of finding the “best compromise” solution.

Numerous MODM models have been proposed in the literature for reaching the best compromise between conflicting objectives [5], [6], [20]. The technique for order of preference by similarity to ideal solution (TOPSIS) method introduced by Hwang and Yoon [6] is a well-known MODM approach. It provides the principle of compromise saying that the chosen solution should have “the shortest distance from the positive ideal solution” and “the farthest distance from the negative ideal solution.” A wide variety of TOPSIS applications has been reported in the literature. Abo-Sinna and Abou-El-Enien [1] applied TOPSIS to large-scale multiple objective programming problems involving fuzzy parameters. Jadidi et al. [7] extended the version of the TOPSIS method proposed in [1] to solve the multi-objective supplier selection problem under price breaks using multi-objective mixed integer linear programming. Lin and Yeh [13] considered solving stochastic computer network optimization problems by employing the TOPSIS and genetic algorithms. Khalili-Damghani et al. [8] used a TOPSIS method to confine the objective dimension space of real-life large-scale multi-objective multi-period project selection problems.

In this paper the basic principle of compromise of TOPSIS is applied for solving the bipolar max-T_p equation constrained multi-objective optimization problem. It shows that such a problem can be reformulated into a 0-1 integer program and then solved taking advantage of well developed techniques in integer optimization. The rest of this paper is organized as follows. In Section 2, the compromise solution approach for solving the bipolar max-T_p equation constrained multi-objective optimization problem (2) is presented. Some critical features associated with the feasible domain and optimal solutions of the bipolar max-T_p equation constrained optimization problem are studied in Section 3. A numerical example is included in Section 4 to illustrate the integer optimization based compromise solution procedure. This paper is concluded in Section 5.

2. A COMPROMISE SOLUTION APPROACH

To solve the bipolar max-$\mathcal{T}_p$ equation constrained multi-objective optimization problem (1), we adopt the principle of compromise, i.e., the chosen solution should have “the shortest distance from the positive ideal solution” and “the farthest distance from the negative ideal solution.” Let $X \triangleq \{\mathrm{x} \in [0,1]^n \mid A^+ \circ \mathrm{x} \vee A^- \circ \bar{\mathrm{x}} = \mathrm{b}\}$ be the feasible domain and I, J be two index sets. For each $j \in J$, $f_j(\mathrm{x})$ is an objective function to be maximized. Similarly, for each $i \in I$, $f_i(\mathrm{x})$ is an objective function to be minimized. To define the positive ideal solution and negative ideal solution of the problem (1), for each $k,\ k = 1, 2, \cdots, K$, we consider

$$f_k^* = \begin{cases} \max_{\mathbf{x}\in X} f_k(\mathbf{x}) & \text{if } k \in J, \\ \min_{\mathbf{x}\in X} f_k(\mathbf{x}) & \text{if } k \in I, \end{cases} \tag{3}$$

and

$$f_k^- = \begin{cases} \min_{\mathbf{x}\in X} f_k(\mathbf{x}) & \text{if } k \in J, \\ \max_{\mathbf{x}\in X} f_k(\mathbf{x}) & \text{if } k \in I. \end{cases} \tag{4}$$

Let $f^* \triangleq (f_1^*, f_2^*, \cdots, f_K^*)^T \in \mathbf{R}^k$ be the solution vector of equation (3) which consists of individual best feasible solutions for all objectives. f^* is then called the positive ideal solution (PIS). Similarly, let $f^- \triangleq (f_1^-, f_2^-, \cdots, f_K^-)^T \in \mathbf{R}^k$ be the solution vector of equation (4) which consists of individual worst feasible solutions for all objectives. f^- is then called the negative ideal solution (NIS).

To measure the distances from PIS and NIS to all objectives, the Minkowski's L_p-metric is employed, i.e., the distance between two points $f_k(\mathbf{x})$ and f_k^* (or f_k^-), $k = 1, 2, \cdots, K$, is defined by the L_p-norm with $p \geq 1$. Moreover, because of the incommensurability among objectives, the component distance from PIS or NIS for each objective is normalized. The following distance functions are then considered:

$$d_p^{PIS}(\mathbf{x}) = \left\{ \sum_{j\in J} w_j^p [\frac{f_j^* - f_j(\mathbf{x})}{f_j^* - f_j^-}]^p + \sum_{i\in I} w_i^p [\frac{f_i(\mathbf{x}) - f_i^*}{f_i^- - f_i^*}]^p \right\}^{1/p}$$

and

$$d_p^{NIS}(\mathbf{x}) = \left\{ \sum_{j\in J} w_j^p [\frac{f_j(\mathbf{x}) - f_j^-}{f_j^* - f_j^-}]^p + \sum_{i\in I} w_i^p [\frac{f_i^- - f_i(\mathbf{x})}{f_i^- - f_i^*}]^p \right\}^{1/p}$$

where d_p^{PIS} and d_p^{NIS} are the distances from the PIS and NIS to all objectives, respectively, $w_k \in [0, 1], k = 1, 2, \cdots, K$, is the relative importance (weight) of objective function k, and $p = 1, 2, \cdots, \infty$ is the parameter of norm functions.

To consider the objectives of "minimize the distance from PIS or d_p^{PIS}" and "maximize the distance from NIS or d_p^{NIS}" instead of the original K objectives in problem (1), we have the following bi-objective programming problem:

$$\begin{aligned} \min \quad & d_p^{PIS}(\mathbf{x}) \\ \max \quad & d_p^{NIS}(\mathbf{x}) \\ \text{s.t.} \quad & A^+ \circ \mathbf{x} \vee A^- \circ \bar{\mathbf{x}} = \mathbf{b}, \\ & \mathbf{x} \in [0,1]^n. \end{aligned} \tag{5}$$

Among all p values, the case of $p = 1$ is operationally and practically important, which provides better credibility than others in the measuring concept and emphasizes the sum of individual distances (regrets for d_p^{PIS} and rewards for d_p^{NIS}) in the utility concept [4]. Our work adopts $p = 1$ for finding the compromise solution to the bipolar max-$\mathcal{T}_p$ equation constrained multi-objective optimization problem (2). For the rest of the paper, $p = 1$ is chosen, although other values may be applicable.

Lemma 1. The compromise solution of problem (1) can be obtained by solving the following bipolar max-$\mathcal{T}_p$ equation constrained optimization problem:

$$\begin{aligned} \min \quad & d_1^{PIS} \\ \text{s.t.} \quad & A^+ \circ \mathbf{x} \vee A^- \circ \bar{\mathbf{x}} = \mathbf{b}, \\ & \mathbf{x} \in [0,1]^n, \end{aligned} \tag{6}$$

or

$$\begin{aligned} \max \quad & d_1^{NIS} \\ \text{s.t.} \quad & A^+ \circ \mathbf{x} \vee A^- \circ \bar{\mathbf{x}} = \mathbf{b}, \\ & \mathbf{x} \in [0,1]^n. \end{aligned} \tag{7}$$

Proof: Since $d_p^{PIS} = 1 - d_p^{NIS}$ for the case of $p = 1$, "min d_1^{PIS}" and "max d_1^{NIS}" are subjected to the same system of max-$\mathcal{T}_p$ equation constraints and have the same solution whether the weights of the objectives are the same or not. Thus, solving the bi-objective programming problem (5) is equivalent to solving either problem (6) or problem (7). Therefore, the compromise solution of problem (1) can be obtained by solving a bipolar max-$\mathcal{T}_p$ equation constrained optimization problem.

In the implementation of TOPSIS for solving the bipolar max-T_p equation constrained multi-objective optimization problem (2), we face the challenge of solving the bipolar max- T_p equation constrained optimization problems (3), (4), (6) or (7). Some important properties associated with the feasible domain and optimal solutions of the the bipolar max- T_p equation constrained optimization problem are studied in Section 3. An integer optimization based technique is applied to reformulated the bipolar max- T_p equation constrained optimization problem into a 0-1 integer programming problem.

3. BIPOLAR MAX-T$_p$ EQUATION CONSTRAINED OPTIMIZATION PROBLEMS

Let $M = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$ be two index sets. A system of bipolar max-$\mathcal{T}_p$ equations $A^+ \circ \mathrm{x} \vee A^- \circ \bar{\mathrm{x}} = \mathrm{b}$ can be reformulated as

$$\max_{j\in N} \max\{\mathcal{T}_p(a_{ij}^+, x_j), \mathcal{T}_p(a_{ij}^-, 1 - x_j)\} = b_i,\ \ i \in M.$$

The system of bipolar max-$\mathcal{T}_p$ equations $A^+ \circ \mathrm{x} \vee A^- \circ \bar{\mathrm{x}} = \mathrm{b}$ is called consistent if its solution set $S(A^+, A^-, b)$ is nonempty. Otherwise, it is said to be inconsistent. It is clear that a vector $\mathrm{x} \in S(A^+, A^-, \mathrm{b})$ if and only if $\max\{\mathcal{T}_p(a_{ij}^+, x_j), \mathcal{T}_p(a_{ij}^-, 1 - x_j)\} \leq b_i$, for every $i \in M$, and $j \in N$, and there exists an index $j_i \in N$ for each $i \in M$ such that $\max\{\mathcal{T}_p(a_{ij_i}^+, x_{j_i}), \mathcal{T}_p(a_{ij_i}^-, 1 - x_{j_i})\} = b_i$. To investigate the solution properties to the system of bipolar max-$\mathcal{T}_p$ equations, the inequality of the form $\max\{\mathcal{T}_p(a^+, x), \mathcal{T}_p(a^-, 1 - x)\} \leq b$ for any $a^+, a^-, b \in [0, 1]$ is studied in Lemma 2.

Lemma 2 For any $a^+, a^-, b \in [0, 1]$, the inequality

$$\max\{\mathcal{T}_p(a^+, x), \mathcal{T}_p(a^-, 1 - x)\} \leq b$$

holds if and only if

$$\frac{a^- - b}{a^-} \leq x \leq \frac{b}{a^+}. \tag{8}$$

Proof: Without restricting $x \in [0, 1]$, we have $\max\{a^+x,\ a^-(1 - x)\} \leq b$ if and only if $\frac{a^- - b}{a^-} \leq x \leq \frac{b}{a^+}$. Hence, we have

$$\max\{\mathcal{T}_p(a^+, x), \mathcal{T}_p(a^-, 1 - x)\} \leq b$$

if and only if

$$\frac{a^- - b}{a^-} \leq x \leq \frac{b}{a^+}.$$

Lemma 2 provides the information of lower and upper bounds on the solutions to a system of max-$\mathcal{T}_p$ equations. Denote $\check{\mathrm{x}} = (\check{x}_1, \check{x}_2, \cdots, \check{x}_n)^T$ with

$$\check{x}_j = \max_{i\in M} \frac{a_{ij}^- - b_i}{a_{ij}^-}, \forall\ j \in N,$$

and $\hat{\mathrm{x}} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)^T$ with

$$\hat{x}_j = \min_{i \in M} \frac{b_i}{a_{ij}^+}, \forall\ j \in N.$$

It is clear that if $S(A^+, A^-, \mathrm{b}) \neq \emptyset$, then $\check{\mathrm{x}} \leq \hat{\mathrm{x}}$. Moreover, if $\mathrm{x} \in S(A^+, A^-, \mathrm{b})$, then $\check{\mathrm{x}} \leq \mathrm{x} \leq \hat{\mathrm{x}}$, i.e., the vectors $\check{\mathrm{x}}$ and $\hat{\mathrm{x}}$ serve the lower and upper bounds of the solutions to $A^+ \circ \mathrm{x} \vee A^- \circ \bar{\mathrm{x}} = \mathrm{b}$, respectively. It should be noticed that $\check{\mathrm{x}}$ and $\hat{\mathrm{x}}$ may not necessarily be solutions to $A^+ \circ \mathrm{x} \vee A^- \circ \bar{\mathrm{x}} = \mathrm{b}$ when $S(A^+, A^-, \mathrm{b}) \neq \emptyset$. Even if $\check{\mathrm{x}}, \hat{\mathrm{x}} \in S(A^+, A^-, \mathrm{b})$, it does not imply that $S(A^+, A^-, \mathrm{b}) = \{\mathrm{x} \mid \check{\mathrm{x}} \leq \mathrm{x} \leq \hat{\mathrm{x}}\}$ [11].

To characterize the properties of the solutions to bipolar max-$\mathcal{T}_p$ equations, we consider the following characteristic matrix $\tilde{Q} = (\tilde{q}_{ij})_{m \times n}$, which includes all the critical information for the equality requirements in $A^+ \circ \mathrm{x} \vee A^- \circ \bar{\mathrm{x}} = \mathrm{b}$:

$$\tilde{q}_{ij} = \begin{cases} \{\check{x}_j\}, & \text{if } \mathcal{T}_p(a_{ij}^-, 1 - \check{x}_j) = b_i \neq \mathcal{T}_p(a_{ij}^+, \hat{x}_j), \\ \{\hat{x}_j\}, & \text{if } \mathcal{T}_p(a_{ij}^-, 1 - \check{x}_j) \neq b_i = \mathcal{T}_p(a_{ij}^+, \hat{x}_j), \\ \{\check{x}_j, \hat{x}_j\}, & \text{if } \mathcal{T}_p(a_{ij}^-, 1 - \check{x}_j) = b_i = \mathcal{T}_p(a_{ij}^+, \hat{x}_j), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Based on the similar argument in [11], we have the following result.

Theorem 1 [11] Let $A^+ \circ \mathrm{x} \vee A^- \circ \bar{\mathrm{x}} = \mathrm{b}$ be a system of bipolar max-$\mathcal{T}_p$ equations. A vector $\mathrm{x} \in [0,1]^n$ is a solution to $A^+ \circ \mathrm{x} \vee A^- \circ \bar{\mathrm{x}} = \mathrm{b}$ if and only if $\check{\mathrm{x}} \leq \mathrm{x} \leq \hat{\mathrm{x}}$ and the induced binary matrix $Q^x = (q_{ij}^x)_{m \times n}$ has no zero rows where

$$q_{ij}^x = \begin{cases} 1, & \text{if } x_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1 characterizes the properties of solutions to a system of bipolar max-$\mathcal{T}_p$ equations. Consider the problem of minimizing a linear objective function $c^T\mathrm{x}$ subject to a system of bipolar max-$\mathcal{T}_p$ equations

$$\begin{aligned} \min \quad & c^T \mathrm{x} \\ \text{s.t.} \quad & A^+ \circ \mathrm{x} \vee A^- \circ \bar{\mathrm{x}} = \mathrm{b}. \end{aligned} \tag{9}$$

Suppose that x^* is an optimal solution to the bipolar max-$\mathcal{T}_p$ equation constrained linear optimization problem (9) and there exists an index $k \in N$ such that $\check{x}_k < x_k^* < \hat{x}_k$. By

Theorem 1, the resulting vector remains feasible if the value of x_k^* is increased to $\hat{x}_k$ or decreased to $\check{x}_k$. Consequently, such a modification can be conducted without increasing the objective value according to the sign of c_k in the objective function. Therefore, such an optimal solution $\mathbf{x}^*$ must exist with $x_j^* = \check{x}_j$ or $x_j^* = \hat{x}_j$ for each $j \in N$. We then have the following result.

Lemma 3 Consider a consistent system of bipolar max-$\mathcal{T}_p$ equations $A^+ \circ \mathbf{x} \vee A^- \circ \bar{\mathbf{x}} = \mathbf{b}$. There exists an optimal solution $\mathbf{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ to the optimization problem (9) such that for each $j \in N$ either $x_j^* = \check{x}_j$ or $x_j^* = \hat{x}_j$.

According to Lemma 3, we consider seeking the optimal solution among those assuming the component values contained only in $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$. Let the associated binary variable u_j for each x_j is defined as

$$u_j = \begin{cases} 0, \text{if } x_j = \check{x}_j, \\ 1, \text{if } x_j = \hat{x}_j, \end{cases} \quad \forall\ j \in N.$$

The decision vector $\mathbf{x}$ of $A^+ \circ \mathbf{x} \vee A^- \circ \bar{\mathbf{x}} = \mathbf{b}$ can then be represented by

$$\mathbf{x} = \check{\mathbf{x}} + V\mathbf{u}.$$

where $V = diag(\hat{\mathbf{x}} - \check{\mathbf{x}})$.

For the characteristic matrix $\tilde{Q}$, we define

$$q_{ij}^+ = \begin{cases} 1, & \text{if } \hat{x}_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$q_{ij}^- = \begin{cases} 1, & \text{if } \check{x}_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\check{\mathbf{x}} \leq \check{\mathbf{x}} + V\mathbf{u} \leq \hat{\mathbf{x}}$ for any $\mathbf{u} \in \{0,1\}^n$. By Theorem 1, $\check{\mathbf{x}} + V\mathbf{u}$ induces a binary matrix $Q^+\text{diag}(\mathbf{u}) + Q^-\text{diag}(\mathbf{e} - \mathbf{u})$,where e is a vector of all ones. As long as this matrix has no zero rows, i.e.,

$$Q^+\mathbf{u} + Q^-(\mathbf{e} - \mathbf{u}) = (Q^+ - Q^-)\mathbf{u} + Q^-\mathbf{e} \geq \mathbf{e},$$

$\check{\mathbf{x}} + V\mathbf{u}$ is a solution to $A^+ \circ \mathbf{x} \vee A^- \circ \bar{\mathbf{x}} = \mathbf{b}$. We then have the following result.

Lemma 4 Let $A^+ \circ \mathbf{x} \vee A^- \circ \bar{\mathbf{x}} = \mathbf{b}$ be a system of bipolar max-$\mathcal{T}_p$ equations. A binary vector $\mathbf{u} \in \{0,1\}^n$ induces a solution $\mathbf{x} = \check{\mathbf{x}} + V\mathbf{u}$ to $A^+ \circ \mathbf{x} \vee A^- \circ \bar{\mathbf{x}} = \mathbf{b}$ if and only if $(Q^+ - Q^-)\mathbf{u} + Q^-\mathbf{e} \geq \mathbf{e}$ where $\mathbf{e}$ is the vector of all ones, $Q^+ = (q_{ij}^+)_{m\times n}$ and $Q^- = (q_{ij}^-)_{m\times n}$.

Consequently, the bipolar max-$\mathcal{T}_p$ equation constrained optimization problem (9) can be reformulated to the 0-1 integer optimization problem

$$\begin{array}{ll} \min & c^T\check{\mathbf{x}} + \mathbf{c}^T V\mathbf{u} \\ \text{s.t.} & (Q^+ - Q^-)\mathbf{u} + Q^-\mathbf{e} \geq \mathbf{e} \\ & \mathbf{u} \in \{0,1\}^n. \end{array} \tag{10}$$

Based on the above discussion, an integer programming based TOPSIS algorithm for finding the compromise solution of the bipolar max-T_p equation constrained multi-objective opti-mization problem (2) can be organized as below.

An Algorithm

Step 1. Decision maker provides the relative importance w_k of the K objective functions. (There are various methods including the eigenvector, weighted least square, entropy and LINMAP methods for assessing w_k [6].)

Step 2. Determine the positive ideal solution (f^*) by solving equation (3).

Step 2.1. Construct the associated 0-1 integer programming problem of (3) by Lemma 4 and set $\mathcal{C} = 1$.

Step 2.2. Solve the associated 0-1 integer programming problem using a commercial solver, e.g. CPLEX.

Step 3. If $\mathcal{C} = 1$, then go to Step 4; else if $\mathcal{C} = 2$, then go to Step 5. Otherwise, output the obtained solution as the compromise solution of (1) and go to Step 6.

Step 4. Determine the negative ideal solution (f^-) by solving equation (4).

Step 4.1. Construct the associated 0-1 integer programming problem of (4) by Lemma 4 and set $\mathcal{C} = 2$.

Step 4.2. Go to Step 2.2.

Step 5. Substitute the positive ideal solution and the negative ideal solution obtained in Steps 2 and 4 into problem (6), construct its associated 0-1 integer programming problem by Lemma 4 and set $\mathcal{C} = 3$. Go to Step 2.2.

Step 6. If the compromise solution of (1) obtained by the integer programming based TOPSIS is satisfied, stop. Otherwise, the decision maker may like to change w_k. Then, go back to Step 1. The solution procedure is then repeated.

4. A NUMERICAL EXAMPLE

In this section, a numerical example is provided to illustrate the proposed integer programming based TOPSIS for solving the system of bipolar max-T_p equation constrained multi-objective optimization problem.

Example 1 Consider the bipolar max-T_p equation constrained multi-objective optimization problem.

$$\begin{aligned}
\min\quad & f_1(\mathrm{x}) = x_1 + x_2 + 2x_3 + x_4 \\
\min\quad & f_2(\mathrm{x}) = 2x_1 + x_2 - 4x_3 - x_4 \\
\text{s.t.}\quad & \begin{pmatrix} 0.6 & 0.48 & 0.48 & 0.56 \\ 0.48 & 0.6 & 0.5 & 0.64 \\ 0.96 & 1 & 0.72 & 0.8 \\ 0.56 & 0.64 & 0.625 & 0.5 \end{pmatrix} \circ \mathrm{x} \vee \begin{pmatrix} 0.56 & 0.6 & 0.625 & 0.6 \\ 0.6 & 0.96 & 0.6 & 0.96 \\ 0.96 & 0.8 & 0.96 & 1 \\ 0.625 & 1 & 0.64 & 0.8 \end{pmatrix} \circ \bar{\mathrm{x}} = \begin{pmatrix} 0.42 \\ 0.48 \\ 0.72 \\ 0.5 \end{pmatrix}, \\
& x_j \in [0,1],\ j = 1,2,3,4.
\end{aligned} \tag{11}$$

Let $\mathrm{x} \in X \triangleq \left\{ \mathrm{x} \in [0,1]^4 \mid \begin{pmatrix} 0.6 & 0.48 & 0.48 & 0.56 \\ 0.48 & 0.6 & 0.5 & 0.64 \\ 0.96 & 1 & 0.72 & 0.8 \\ 0.56 & 0.64 & 0.625 & 0.5 \end{pmatrix} \circ \mathrm{x} \vee \begin{pmatrix} 0.56 & 0.6 & 0.625 & 0.6 \\ 0.6 & 0.96 & 0.6 & 0.96 \\ 0.96 & 0.8 & 0.96 & 1 \\ 0.625 & 1 & 0.64 & 0.8 \end{pmatrix} \circ \bar{\mathrm{x}} = \begin{pmatrix} 0.42 \\ 0.48 \\ 0.72 \\ 0.5 \end{pmatrix} \right\}.$

Applying the basic principle of compromise of TOPSIS, problem (11) can be reduced to the following bipolar max-T_p equation constrained optimization problem:

$$\min_{\mathrm{x} \in X} \quad d_1^{PIS}(\mathrm{x}) = w_1 \left[\frac{f_1(\mathrm{x}) - f_1^*}{f_1^- - f_1^*} \right] + w_2 \left[\frac{f_2(\mathrm{x}) - f_2^*}{f_2^- - f_2^*} \right] \tag{12}$$

where

$$f_1^* = \min_{\mathrm{x} \in X} f_1(\mathrm{x}), \tag{13}$$

$$f_2^* = \min_{\mathrm{x} \in X} f_2(\mathrm{x}), \tag{14}$$

$$f_1^- = \max_{\mathrm{x} \in X} f_1(\mathrm{x}), \tag{15}$$

$$f_2^- = \max_{\mathrm{x} \in X} f_2(\mathrm{x}), \tag{16}$$

Consider the bipolar max-T_p equation constrained optimization problem (16):

$$\max_{\mathrm{x} \in X} \quad 2x_1 + x_2 - 4x_3 - x_4$$

The lower and upper bounds can be calculated as

$$\check{\mathrm{x}} = (0.25, 0.5, 0.25, 0.5)^T, \hat{\mathrm{x}} = (0.7, 0.72, 0.8, 0.75)^T,$$

Respectively, and its characteristic matrix is

$$\tilde{Q} = \begin{pmatrix} \{0.25, 0.7\} & \emptyset & \emptyset & \{0.75\} \\ \emptyset & \{0.5\} & \emptyset & \{0.5, 0.75\} \\ \{0.25\} & \{0.72\} & \{0.25\} & \emptyset \\ \emptyset & \{0.5\} & \{0.8\} & \emptyset \end{pmatrix}.$$

Subsequently, the two 0-1 characteristic matrices can be constructed, respectively, as

$$Q^{+} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, Q^{-} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

According to Lemma 4, the constrained bipolar max-T_p equation constrained optimization problem (16) is equivalent to the problem:

$$\begin{aligned} \max \quad & z_{\mathbf{u}} = -0.5 + 0.9u_1 + 0.22u_2 - 2.2u_3 - 0.25u_4 \\ \text{s.t.} \quad & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \\ & u_j \in \{0, 1\}, \ j = 1, 2, 3, 4. \end{aligned}$$

This 0-1 integer linear optimization problem has the optimal solution

$$\mathbf{u}^* = (1, 1, 0, 0)^T,$$

with the optimal objective value $z^*_{\mathbf{u}} = 0.62$. The corresponding optimal solution to the problem (16) is

$$\mathbf{x}^* = (0.7, 0.72, 0.25, 0.5)^T$$

with the optimal objective value $f_2^- = 0.62$.

The solutions of problems (13)-(16) can be obtained in an analogous manner and are shown in Table 1.

Table 1: The solutions of problems (13)-(16).

	f_1	f_2	x_1	x_2	x_3	x_4
$\min_{\mathbf{x}\in X} f_1(\mathbf{x})$	$f_1^* = 1.75$		0	0.8	0.7	0.5
$\min_{\mathbf{x}\in X} f_2(\mathbf{x})$		$f_2^* = -2.95$	0.7	0.72	0.8	0.75
$\max_{\mathbf{x}\in X} f_1(\mathbf{x})$	$f_1^- = 3.77$		0	0.8	0.5	0
$\max_{\mathbf{x}\in X} f_2(\mathbf{x})$		$f_2^- = 0.62$	0	0.8	0.5	0

Substituting the results in Table 1 into the problem (12) with $w_1 = w_2 = \frac{1}{2}$, we have the problem:

$$\min_{\mathbf{x}\in X} 0.5276x_1 + 0.3876x_2 - 0.0582x_3 + 0.1075x_4 - 0.0200$$

Solving the above problem by Lemma 4, a compromise solution of the bipolar max-$\mathcal{T}_p$ equation constrained multi-objective optimization problem (11) can be obtained as

$$\mathbf{x}^* = (0.25, 0.5, 0.75, 0.5)^T.$$

5. CONCLUSIONS

This paper studies the compromise solution to the bipolar max-T_p equation constrained multi-objective optimization problem. Some important properties associated with the bipolar max-T_p equation constrained optimization problem are studied. Since the feasible domain of the bipolar max-T_p equation constrained optimization problem is non-convex, traditional mathe-matical programming techniques may have difficulty in yielding efficient solutions for such an optimization problem. An integer optimization based TOPSIS is proposed to reformulated the bipolar max-T_p equation constrained optimization problem into a 0-1 integer programming problem. Such optimization problems can be practically solved using a commercial solver, e.g. CPLEX. Taking advantage of the well developed techniques in integer optimization, it is expected that wider applications of the proposed method for solving the bipolar max-T equation constrained multi-objective optimization problems are foreseeable in the future.

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