STATISTICAL DISTRIBUTIONS OF DISCRETE WALSH HADAMARD TRANSFORM COEFFICIENTS OF NATURAL IMAGES

Vijay Kumar Nath and Deepika Hazarika

Department of Electronics and Communication Engineering, School of Engineering,
Tezpur University (A Central University), Napaam, Tezpur, Assam, India.
vknath@tezu.ernet.in and deepika@tezu.ernet.in

ABSTRACT

For low bit rate applications, the discrete Walsh Hadamard transform (WHT) shows almost comparable results when compared to the popular discrete Cosine transform (DCT) in terms of compression efficiency, peak signal to noise ratio (PSNR) and visual results. The discrete WHT is a best choice which compromises between the computational complexity and compression efficiency. The great advantage of the discrete WHT is its relatively very low computational complexity when compared to DCT. However there is no definitive study reported in literature regarding the statistical distributions of discrete WHT coefficients of natural images. This study performs a χ^2 *goodness of fit test to determine the distribution that best fits the discrete WHT coefficients. The simulation results show that the distribution of a majority of the significant AC coefficients can be modelled by the Generalized Gaussian distribution. The knowledge of the appropriate distribution helps in design of optimal quantizers that may lead to minimum distortion and hence achieve optimal coding efficiency.*

KEYWORDS

Image Compression, Discrete WHT, χ^2 *test, Statistical analysis*

1. INTRODUCTION

DCT [1] has been very popularly used in many image and video coding standards like JPEG, MPEG1, MPEG2, H.261 [2,3]. But the computational complexity of the DCT is relatively very high which may not be preferred in several real time applications.

The discrete WHT [4,5,6,18] is a orthogonal non sinusoidal transform which shows almost comparable performance in image compression applications when compared to DCT. The discrete WHT is popular for its very low computational complexity. The discrete WHT is considered as the best option whenever there is a compromise between the energy compaction capability and computational complexity. However till now, no study has been reported in the literature, on the distribution of discrete WHT coefficients of the natural images. In contrast, there is a huge amount of studies reported in the literature dealing with the distributions of 2D DCT coefficients of natural images [7,8,9,10].

In [7], considering Gaussian, Laplacian, Gamma and Rayleigh distributions as probable models, Reininger and Gibson used Kolomogrov-Smirnov (KS) test and shown that DC coefficients can be well approximated by Gaussian distribution and AC coefficients can be well approximated by Laplacian distribution. In [8], the authors concluded that no particular density function can be used for each of the coefficients but Laplacian fits majority of the coefficients and when all

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the coefficients are lumped into one density function, the Cauchy distribution provides the best fit. In [9], Muller found that the Generalized Gaussian distribution best approximates the statistics of the 2D DCT coefficients. Smoot and Reeve [19] study the statistics of the DCT coefficients of the differential signal obtained after motion estimation.
They observe that the statistics are best approximated by the Laplacian distribution.

In this paper, considering Gaussian, Laplacian, Gamma, Generalized Gaussian and Cauchy distributions as probable models [11,12], we study the statistical distribution using χ^2 Goodness of Fit test [13]. The model distribution that gives the minimum χ^2 statistic is chosen as the best fit. The χ^2 test of fit showed that Generalized Gaussian distribution best models the statistics of discrete WHT coefficients of natural images. These results can be used to design optimal quantizers for discrete WHT coefficients. The organization of the paper is as follows. Section 2 provides the introduction to discrete WHT. χ^2 Goodness of Fit test is explained in Section 3. Section 4 describes the model statistical distributions. Experimental results are presented in Section 5 and the paper is concluded in Section 6.

2. DISCRETE WALSH HADAMARD TRANSFORM

The basis functions of discrete WHT [4] are not sinusoids unlike Fourier and Cosine transforms. The basis functions are based on rectangular waveforms with peaks of ± 1. The N-point 1-D discrete Walsh Hadamard transform of a signal $x(n)$ is given as

$$V(z) = \frac{1}{N}\sum_{m=0}^{N-1} x(m) y_z(m), \quad z = 0,1,.............,N-1 \tag{1}$$

where $y_z(m)$ is the Walsh function of order z and is defined recursively as:

$$y_z(m) = 0 \text{ for } m < 0 \text{ and } m > N-1$$
$$y_0(m) = 1 \text{ for } m = 0,1......,N-1$$
$$y_{2z}(m) = y_z(2m) + (-1)^z y_z(2m-1)$$
$$y_{2z+1}(m) = y_z(2m) - (-1)^z y_z(2m-1)$$

for $z = 0,1.........N-1$.

3. χ^2 GOODNESS OF FIT TEST

χ^2 and KS test [13] are the widely used goodness of fit test. The KS goodness of fit test statistic is a distance measure between the empirical cumulative distribution function (CDF) for a given data set and the given model cumulative distribution function. In contrast to the KS test, the χ^2 goodness of fit test compares the model probability density functions with the empirical data and finds out the distortion by the following equation

$$\chi^2 = \sum_{i=1}^{k_e} \frac{(O_i - E_i)^2}{E_i} \tag{2}$$

where the range of data is partitioned into k_e disjoint and exhaustive bins B_i, $i = 1,2,3,.........,k_e$. $E_i = n_c p_i$ is the expected frequency in bin B_i where $p_i = P(x \in B_i)$ and O_i is the observed frequency in bin B_i. n_c is the total number of data samples. The model probability density function which gives the minimum χ^2 value can be considered as the best fit.
The χ^2 test is preferred over the KS test because for source coding the deviation from an assumed probability density function is more interesting than deviation from a distribution function [9].

4. PROBABILITY DISTRIBUTIONS

We use χ^2 goodness of fit tests considering Gaussian, Laplacian, Gamma, Cauchy and Generalized Gaussian distributions [11,12] as probable models as these distributions are commonly used for statistical modeling of DCT coefficients. The parameters of the distributions were found using the maximum likelihood (ML) method.

4.1. Gaussian Probability Density Function

The Gaussian probability density function is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{3}$$

where μ is the mean and σ^2 is the variance. The ML estimates of μ and σ^2 are given by

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{4}$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 \tag{5}$$

4.2. Laplacian Probability Density Function

The Laplacian probability density function is given by

$$f_X(x) = \frac{1}{2a}\exp\left(-\frac{|x-\mu|}{a}\right) \tag{6}$$

where μ is the mean and a is the scale parameter. The variance is given by $2a^2$. The ML estimate of the parameter a is given by

$$\hat{a} = \frac{1}{N}\sum_{i=1}^{N}\left|x_i - \hat{\mu}\right| \tag{7}$$

where μ is estimated using (4).

4.3. Gamma Probability Density Function

The Gamma probability density function is given by [14]

$$f_X(x) = \frac{\sqrt[4]{3}}{\sqrt{8\pi\sigma|x-\mu|}}\exp\left(-\frac{\sqrt{3}|x-\mu|}{2\sigma}\right) \tag{8}$$

The parameters μ and σ are estimated using (4) and (5) respectively.

4.4. Generalized Gaussian Probability Density Function

The Generalized Gaussian probability density function [11,12,15] is given by

$$f_X(x) = \frac{\beta}{2\alpha\Gamma\left(\frac{1}{\beta}\right)}\exp\left(-\frac{|x|}{\alpha}\right)^{\beta} \tag{9}$$

where $\Gamma(.)$ is the Gamma function given by

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, z > 0$$

and α, β respectively are known as the scale parameter and the shape parameter. For the special cases $\beta = 2$ or $\beta = 1$, the generalized Gaussian pdf becomes a Gaussian or a Laplacian pdf respectively. The ML estimation of the parameters α and β can be obtained as follows:

The shape parameter $\hat{\beta}$ is the solution of the equation

$$1+\frac{\psi\left(\frac{1}{\beta}\right)}{\beta}-\frac{\sum_{i=1}^{N}|x_i|^{\beta}\log|x_i|}{\sum_{i=1}^{N}|x_i|^{\beta}}+\frac{\log\left(\frac{\beta}{N}\sum_{i=1}^{N}|x_i|^{\beta}\right)}{\beta}=0 \qquad (10)$$

where $\psi(.)$ is the digamma function given by

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$$

The ML estimate of α (for known ML estimate $\hat{\beta}$) is given by

$$\hat{\alpha}=\left(\frac{\hat{\beta}}{N}\sum_{i=1}^{N}|x_i|^{\hat{\beta}}\right)^{\frac{1}{\hat{\beta}}} \qquad (11)$$

where N is number of observations.

The $\hat{\beta}$ is determined using the Newton Raphson iterative procedure with the initial guess from the moment based method described in [16].

4.4. Cauchy Probability Density Function

The Cauchy probability density function [11,12,17] is given by

$$f_X(x)=\frac{1}{\pi}\left[\frac{\gamma}{(x-x_0)^2+\gamma^2}\right] \qquad (12)$$

where x_0 is the location parameter and γ is the scale parameter. For the Cauchy distribution with sample size N, the likelihood function $L(x_1,.......x_N;x_0,\gamma)$ is given by

$$L(x_1,.......x_N;x_0,\gamma)=\prod_{i=1}^{N}\left[\frac{1}{\pi\gamma\left\{1+(x_i-x_0/\gamma)^2\right\}}\right] \qquad (13)$$

The logarithm of the likelihood is given by

$$\log L=-N\log\pi-N\log\gamma-\sum_{i=1}^{N}\log\left\{1+\left(\frac{x_i-x_0}{\gamma}\right)^2\right\} \qquad (14)$$

Maximizing the log likelihood function with respect to y_0 and γ gives:

$$\sum_{i=1}^{N}\left(\frac{x_i-x_0}{\gamma}\right)\Big/\left\{1+\left(\frac{x_i-x_0}{\gamma}\right)^2\right\}=0 \qquad (15)$$

$$\sum_{i=1}^{N}\left\{1+\left(\frac{x_i-x_0}{\gamma}\right)^2\right\}^{-1}-\frac{N}{2}=0 \qquad (16)$$

We use Newton Raphson iterative procedure to solve (15) and (16) for x_0 and γ. The median is used as an initial value for x_0 as described in [17].

5. EXPERIMENTAL RESULTS

The χ^2 goodness of fit performance was evaluated using discrete WHT coefficients (8x8 block size) for fifteen natural test images having different characteristics but here we report only for popular Lena, Barbara, Aerial, House, Boats and Mandrill image. The parameters of all distributions were found using maximum likelihood method. The model distribution which provides the minimum χ^2 statistic is considered to be the best fit under the χ^2 criterion. The nine AC coefficients C_{10}, C_{11}, C_{01}, C_{20}, C_{02}, C_{12}, C_{21}, C_{03} and C_{30} used in the experiments were chosen because they usually have the most effect on the image quality.

Table 1. χ^2 statistics for a few discrete WHT coefficients of Lena (512x512) image

Coefficient	Gaussian	Laplacian	Gamma	Gen. Gaussian	Cauchy
C_{10}	5.94×10^8	8165	87	**32**	115
C_{11}	4.36×10^8	5740	103	**22**	205
C_{01}	4.61×10^7	954	60	**43**	296
C_{20}	4.23×10^7	2216	55	**45**	130
C_{02}	1.02×10^6	639	**36**	106	333
C_{12}	1.25×10^6	1071	42	**27**	134
C_{21}	1.42×10^6	1156	**25**	45	176
C_{03}	6.38×10^6	4292	168	**88**	163
C_{30}	1.61×10^{12}	20136	141	**42**	88

Table 2. χ^2 statistics for a few discrete WHT coefficients of Barbara (512x512) image

Coefficient	Gaussian	Laplacian	Gamma	Gen. Gaussian	Cauchy
C_{10}	1.00×10^{10}	3184	49	**39**	126
C_{11}	1.12×10^9	2060	**14**	18	162
C_{01}	1.47×10^4	365	35	**22**	274
C_{20}	5.48×10^4	200	122	**21**	196
C_{02}	1.53×10^8	380	30	**9**	197
C_{12}	1.67×10^6	1811	28	**26**	183
C_{21}	2.04×10^7	223	69	**19**	245
C_{03}	1.00×10^5	543	**19**	64	242
C_{30}	2.59×10^5	16608	34	**8**	147

Table 3. χ^2 statistics for a few discrete WHT coefficients of Aerial (256x256) image

Coefficient	Gaussian	Laplacian	Gamma	Gen. Gaussian	Cauchy
C_{10}	144	15	120	**14**	124
C_{11}	51	18	204	**12**	151
C_{01}	243	19	72	**10**	89
C_{20}	145	7	148	**6**	124
C_{02}	76	11	130	**9**	129
C_{12}	198	8	175	**6**	134
C_{21}	72	7	193	**3**	133
C_{03}	5726	28	35	**7**	79
C_{30}	286	42	79	**27**	99

Table 4. χ^2 statistics for a few discrete WHT coefficients of House (256x256) image

Coefficient	Gaussian	Laplacian	Gamma	Gen. Gaussian	Cauchy
C_{10}	45678	1138	76	**38**	214
C_{11}	5.69x10^7	29566	101	**20**	40
C_{01}	1.42x10^5	1302	124	**112**	167
C_{20}	4903	952	152	**118**	169
C_{02}	8.18x10^6	2331	101	118	**48**
C_{12}	9.28x10^5	3758	51	**13**	26
C_{21}	1.34x10^6	688	31	**16**	68
C_{03}	1.81x10^{13}	2.33x10^7	295	83	**33**
C_{30}	24159	1738	110	**52**	124

Table 5. χ^2 statistics for a few discrete WHT coefficients of Boats (512x512) image

Coefficient	Gaussian	Laplacian	Gamma	Gen. Gaussian	Cauchy
C_{10}	2.19x10^5	741	43	**17**	142
C_{11}	8.60x10^4	636	65	**12**	143
C_{01}	7.48x10^8	3289	47	**34**	140
C_{20}	2.68x10^8	384	48	**25**	202
C_{02}	1.39x10^{10}	1551	66	**33**	155
C_{12}	1.15x10^7	698	13	**5**	171
C_{21}	6.81x10^5	450	**14**	23	189
C_{03}	1.91x10^{13}	6.14x10^4	128	**13**	146
C_{30}	54037	432	**21**	36	312

Table 6. χ^2 statistics for a few discrete WHT coefficients of Mandrill (512x512) image

Coefficient	Gaussian	Laplacian	Gamma	Gen. Gaussian	Cauchy
C_{10}	1774	**21**	388	23	398
C_{11}	781	6	248	**5**	415
C_{01}	15427	**35**	573	40	462
C_{20}	843	72	110	**43**	330
C_{02}	12848	**15**	329	20	421
C_{12}	2.31x10^5	4	288	**2**	369
C_{21}	1317	43	234	**24**	364
C_{03}	23950	7	352	**6**	407
C_{30}	2925	79	163	**18**	311

If $N \text{ x } M$ is the size of the image after computation of 2D discrete WHT, then each coefficient will have $\frac{N}{8} \text{x} \frac{M}{8}$ values for the image to be used in KS and χ^2 test. Table 1, 2, 3, 4, 5 and 6 show the χ^2 statistics of discrete WHT coefficients for Lena, Barbara, Aerial, House, Boats and Mandrill images respectively. The result indicates that the χ^2 test statistic is smallest for the

General Gaussian distribution for most of the tested coefficients of all the images. Fig. 1 shows the empirical pdf of the discrete WHT coefficients along with the fitted Gaussian, Laplacian, Gamma, Generalized Gaussian and Cauchy pdfs. From Fig. 1 as well as from the values of the χ^2 statistic, it is clear that the Generalized Gaussian distribution provides a better fit to the empirical distribution then that achieved by the Gaussian, Laplacian, Gamma and Cauchy pdfs.

6. CONCLUSION

In this paper, we perform the χ^2 goodness of fit tests to determine an appropriate statistical distribution that best models the discrete WHT coefficients of images. The simulation results indicate that no single distribution can be used to model the distributions of all AC coefficients for all natural images. However the distribution of a majority of the significant AC coefficients can be approximated by the Generalized Gaussian distribution. The knowledge of the statistical distribution of transform coefficients is very important in the design of optimal quantizer that may lead to minimum distortion and hence achieve optimal coding efficiency.

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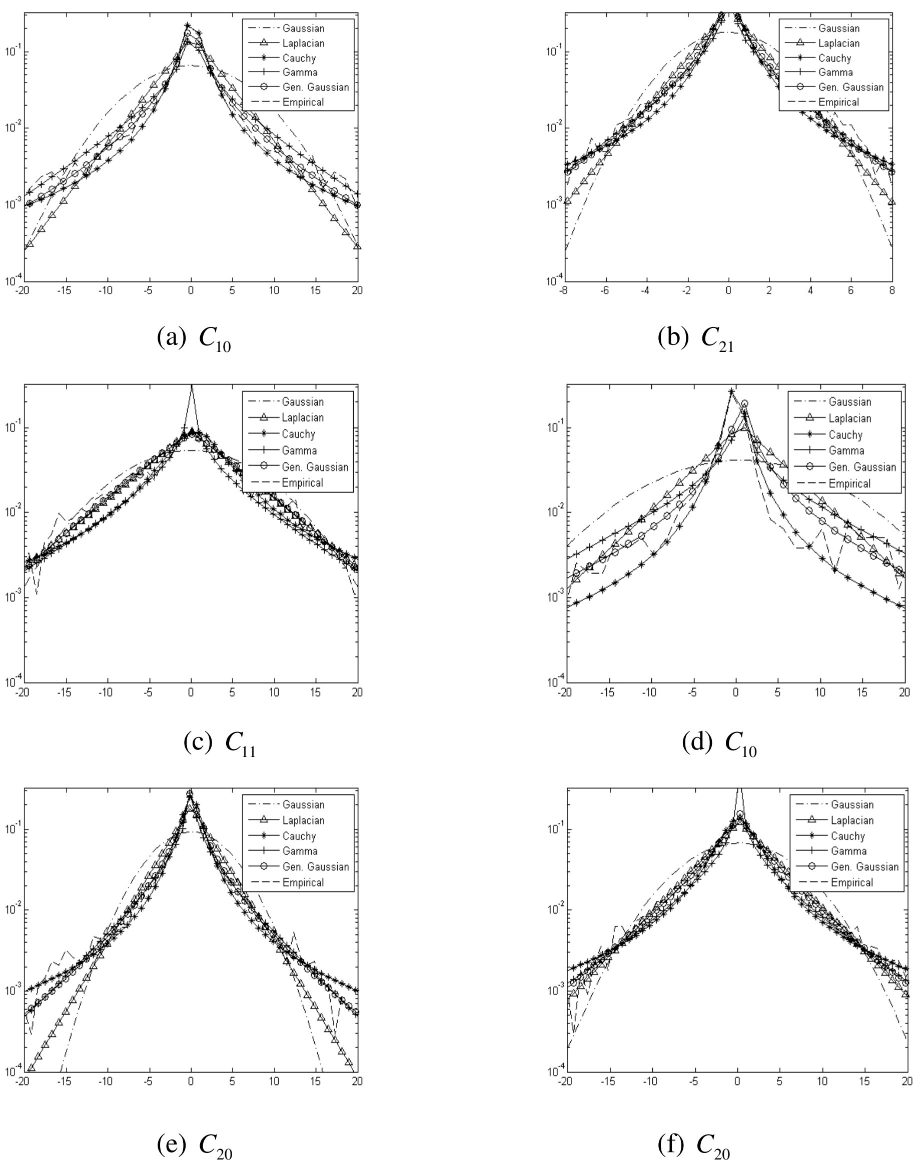

Figure1. Logarithmic Histograms of the discrete WHT coefficients for (a) Lena (b) Barbara (c) Aerial (d) House (e) Boats (f) Mandrill images and the Gaussian, Laplacian, Gamma, Generalized Gaussian and Cauchy pdfs fitted to this histogram in log domain