IMPLICATIVE FILTERS IN SOME FUZZY LOGIC STRUCTURES

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ABSTRACT

In this paper, we attempt to find a characterization of some implicative filters in some fuzzy algebras. Towards this end, firstly, the notion of a De Morgan triples and properties of implicative filters in a de Morgan implicative structure are stated. finally, some interesting with detail of most classical cases of an implicative filters are given.

KEYWORDS

Fuzzy Logic, Implicative Filters, Logic Algebra, Triangular Norms

1. INTRODUCTION

The theory of fuzzy sets and fuzzy logic has been extensively developed in last decades from the pioneer ideas by Zadeh, and many applications have been successfully carried out (see [8,9]).
In fact, there are many situations where fuzzy control applies better than the classical one, mainly in all environments where uncertainty and imprecision are present and high precision in measurements and statements is not required.

A fuzzy implication is the generalization of the classical one to fuzzy logic, much the same way as a t-norm and a t-conorm are generalizations of the classical conjunction and disjunction, respectively.

The main goal of this article is to examine some of implicative filters in a De Morgan algebras by discussing their algebraic properties, in order to obtain a characterization of them.

2. PRELIMINARIES

We briefly introduce some basic notions used in the rest of the work.

2.1. Fuzzy Negations, T-Norms and T-Conorms

Definition 1 (see Fodor and Roubens [1, p.3], Klement et al. [2, Definition11.3]). A decreasing function $N: [0,1] \to [0,1]$ is called a fuzzy negation, if $N(1) = 0$, $N(0) = 1$.

A fuzzy negation N is called:

(i) strict, if it is strictly decreasing and continuous;
(ii) strong, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$;
(iii) non-vanishing, if $N(x) = 0 \Rightarrow x = 1$.

Natarajan Meghanathan et al. (Eds) : CSTY, AI, MaVaS, SIGI, FUZZY - 2019
pp. 127-132, 2019. DOI: 10.5121/csit.2019.91411

Definition 2 (Schweizer and Sklar [3], Klement et al. [2]).

(i) An associative, commutative and increasing operation T: $[0,1]^2 \rightarrow [0,1]$ is called a triangular norm (t-norm, for short), if it has the neutral element equal to 1.
2

(ii) An associative, commutative and increasing operation S: $[0,1]^2 \rightarrow [0,1]$ is called a triangular conorm (t-conorm, for short), if it has the neutral element equal to 0.

2.2. Fuzzy Implications

Definition 3 (see Fodor and Roubens [1, Definition 1.15]). A function I: $[0, 1]^2 \rightarrow [0, 1]$ is called fuzzy implication, if it satisfies, for all x, y, z $\in$ [0, 1], the following conditions:

1. $I(x,z) \geq I(y,z)$ if $x \leq y$,
2. $I(x,y) \leq I(x,z)$ if $y \leq z$,
3. $I(0,0) =1$, $I(1,1) = 1$, $I(1,0) = 0$

Remark Each fuzzy implication I satisfies the following left and right boundary conditions, respectively:

$$I(0,y) = 1,\ y \in [0,1] \ \text{(LB)}$$
$$I(x,1) = 1,\ x \in [0,1] \ \text{(RB)}$$

Therefore, I satisfy also the normality condition: $I(0,1) = 1$ (NC).
Additional properties of fuzzy implications were postulated in many works (see, for example [4,1,5]). The most important of them are presented below.

Definition 4 (see [8, Definition 3.2]). A fuzzy implication I is said to satisfy

(i) the left neutrality property, if $I(1,y) = y$, $y \in [0,1]$ (NP);
(ii) the exchange principle, if for all $x,y,z \in [0,1]$, $I(x,I(y,z))=I(y,I(x,z))$ (EP)
(iii) the identity principle, if for all x in [0,1], $I(x,x)=1$ (IP);
(iiii) the ordering property, if for all $x,y \in [0,1]$, $I(x,y)=1$ if and only if $x=y$ (OP).

3. DE MORGAN FUZZY STRUCTURE

Definition 5 (Klement et al. [2, p.232]). A triple (T,S,N), where T is a t-norm, S is a t-conorm and N is a strict negation, is called a De Morgan triple if for all a, b $\in$ [0, 1],

$$T(a,b) = N^{-1}(S(N(a),N(b))),\ S(a,b) = N^{-1}(T(N(a),N(b))).$$

The set ([0,1],T,S,N) equipped with a de Morgan triple will be called a de Morgan fuzzy structure.

Theorem 1 (Klement et al. [2, p.232]). The two propositions are equivalent:

1. (T,S,N) is a de Morgan triple,
2. N is strong and S is the N−dual of T, i.e., $T(a,b) = N(S(N(a),N(b)))$, for all $a,b \in [0,1]$.
The De Morgan fuzzy structure ([0,1],T,S,N) is said to be distributive if T is distributive over S.
According to (Klement et al. [2, p.34]), the quadruple ([0, 1], $\wedge$, $\vee$, N) is a distributive De Morgan fuzzy structure if and only if, $T = T_M$.

Definition 6 An implicative fuzzy structure L is a de Morgan fuzzy structure equipped with a fuzzy implication, that is L = ([0, 1], T, S, N, I).

In the sequel we use the classical implications I_A, I_B, I_C, I_L for all a,b in [0,1] by :

I_A (a, b) = N (a) $\vee$ b (Kleene-Dienes)
I_B(a,b) = I_A(a,b)$\wedge$(k(a) (Zadeh)
I_C (a, b) = I_A(a, b) $\wedge$ k(a) $\wedge$ k(b) (Willmott)
I_L(a, b) = sup{t $\in$ [0, 1]; a $\wedge$ t $\leq$ b} (the residual implication associated with the t-norm $\wedge$).

Here k(x) = x $\vee$ N(x) is the crispness degree of x.

4. IMPLICATIVE FILTERS IN A DE MORGAN FUZZY STRUCTURE

From now on, L denotes an implicative De Morgan fuzzy structure ([0, 1], $\wedge$, $\vee$, N, I)

Definition 7 A subset F of [0, 1] is said to be an implicative filter of L (or deductive system) if 1$\in$F and (x$\in$F and I(x,y)$\in$F) imply y$\in$F. F is a proper implicative filter if F is a filter and F$\neq$ [0, 1].

Proposition 1 If $(F_i)_{i\in I}$ is a family of filters of L, then $\cup_{i\in I} F_i$ and $\cap_{i\in I} F_i$ are filters of L.

Proposition 2 If a filter F of L contains 0 then F = [0, 1].

Proposition 3 In a De Morgan implicative fuzzy structure with I = I_A, if F is a proper filter of L and x ∈ F then $N(x) \notin F$.

Proposition 4 In an implicative de Morgan structure L, if the implication I satisfy (OP) or (NP), then F = {1} is a proper filter of L.

5. CHARACTERIZATION OF IMPLICATIVE FILTERS IN CLASSICAL LOGIC STRUCTURES

5.1. Logic Model with the Zadeh Operators

Let [0, 1] equipped with Zadeh negation N_C(x) = 1 − x and the t-norm T(x, y) = xy and t-conorm S(x, y) = x + y − xy, and the implication I = I_A . Consider the implicative De Morgan fuzzy structure L equipped with the above operators.

Proposition 5 Let a ∈ [0, 1], the set
F =]0, a] ∪ {1} is a proper filter of L if and only if a < 1/2.

Proposition 6 The set F = [a, 1] is a proper filter of L if and only if a = 1.

5.2. The Case of Lukasiewicz Fuzzy Logic Operators

Let [0, 1] equipped with a Lukasiewicz t-norm and t-conorm given by: T_{LK}(x, y) = max(x+y−1, 0), S_{LK}(x, y) = min(x + y, 1) ([8], Table 2 and 3) and the associated negation N_C(x) = 1 – x.

Proposition 7. With the notations above, ([0, 1], T, S, N) is a de Morgan fuzzy structure.
Remark: This structure is not distributive, example x = 1/2, y = z = 2/3.

The Lukasiewicz fuzzy implication is defined by $I_{LK}(x, y) = \min(1, 1 - x + y)$ ([8], Table 5), since in this case the crispness degree is $k(x) = 1$, this implication coincide with $I_A = I_B = I_C$. I coincides also with the T-residual implication, because:

$I_L(x, y) = \sup\{t \in [0, 1]; x \wedge t \leq y\}, = \sup\{t \in [0, 1]; x + t - 1 \leq y\} = \min(1 + y - x, 1) = I(x, y)$.

Proposition 8. The Lukasiewicz fuzzy implication verify the properties (IP), (NP), (EP) and (OP).

Lemma 1. Let F be a proper filter of L then $[0, 1/2] \cap F = \emptyset$ proof.
Suppose $x \in [0, 1/2] \cap F$, then $I(x, 1/2) = 1 \in F$ so $1/2 \in F$, hence $I(1/2, 0) = 1/2 \in F$ so $0 \in F$ imply $F = [0, 1]$.

Lemma 2 Let F be a proper filter of L then there exist $a \in]1/2, 1]$ such that, $F=]a, 1]$, or $F = [a, 1]$.

Theorem 2. The unique proper filter of Lukasiewicz algebra is $\{1\}$.

5.3 The Case of Godel Fuzzy Logic Operators

In this Case, L, $T(x, y) = \min(x, y)$, $S(x, y) = \max(x, y)$, $I_{GD}(x, y) = 1$ if $x \leq y$ and $I_{GD}(x, y) = y$ if $x > y$, $N(x) = I(x, 0)$. $N(x) = 1$ if $x = 0$ and $N(x) = 0$ elsewhere.

It is obvious that L is an implicative De Morgan fuzzy structure.

Proposition 9. The crispness degree in L is $k(x) =1$ if $x = 0$ x if $x > 0$

Remark. The implication I verify (NP), so $F = \{1\}$ is a proper implicative filter of L.

Proposition 10. If F is a proper implicative filter of L then $x \in F$ imply $N(x) \notin F$.

Proposition 11. The implicative filters of L are the form $[a, 1]$ or $]a,1]$, where $a \in [0, 1]$.

5.4 The Case of Fodor Fuzzy Logic Operators.

We define the t-norm and t-conorm by: $S(x, y) =1$ if $x + y \geq 1$ $\max(x, y)$ otherwise, $T(x, y) = 0$ if $x + y \leq 1$ $\min(x, y)$ otherwise. The so associated negation is $N_C(x) = 1-x$, we obtain a De Morgan fuzzy algebra witch satisfy the LEM, so $I_A = I_B = I_C$ witch is the Fodor's implication defined by $I_{FD}(x, y) = 1$ if $y \geq x$ and $\max(1 - x, y)$ otherwise.
We obtain an implicative De Morgan fuzzy algebra.

Proposition 12. Let F be an implicative filter of L, for all $t \in [1/2, 1]$,
$F_t = \{x \in F; x \geq t\}$ is a filter of L.

In the same way we can show the following proposition;

Proposition 13.
Let F be an implicative filter of L, for all $t \in [1/2, 1]$, $F_t = \{x \in F; x > t\}$ is a filter of L. Now we can characterize all filters in the present algebra.

Proposition 14. The proper implicative filters of L are the sets $]a,1]$ and $[a, 1]$ for $a \in]1/2, 1]$.

5.5 The Case of Drastic Fuzzy Logic Operators

The t-norm and t- conorm defined by: $T(x, y) = 0$ if $x, y < 1$ $\min(x, y)$ otherwise, $S(x, y) = 1$ if $x, y > 0$ $\max(x, y)$ otherwise. This algebra is not a de Morgan algebra, we have $N_S(1/2 \wedge 1/2) = N_S(0) = 1$ but $N_S(1/2) \vee N_S(1/2) = 0$, where the negation $N_S = N_{D1}$ defined by $N_S = 1$ if $x = 0$, and 0 otherwise.

We consider the algebra equipped with $I = I_A$, I_B or I_C. In this algebra the crispness degree is $k(x) = N(x) \vee x = 1$ if $x = 0$ and $k(x) = x$ elsewhere.

Then implication I_A is defined by: $I_A(x, y) = 1$ if $x = 0$, and: $I_A(x, y) = y$ otherwise. In this case the fuzzy algebra L_A satisfy the crispness conservation.

Proposition 15. A subset F of [0, 1] is a filter of L_A if and only if $0 \notin F$, and $1 \in F$, in other words, the only proper filters of L_A are the sets containing 1 and not 0.

5.6 The case of Schweizer - Sklar fuzzy logic operators (see [2], p :104 and [6],example 4.13).

The t-norm and conorm are given by:

$$T_{SS}^2(x, y) = (\max(x^2 + y^2 - 1, 0))^{1/2},\ S_{SS}^2(x, y) = 1 - (\max((1-x)^2 + (1-y)^2 - 1, 0))^{1/2},\ I_{PR}(x, y) = 1 - (\max(x(1 + xy2 - 2y), 0))^{1/2},\ N_C(x) = 1 - x.$$

For all $x \in L$, $k(x) = 1$, so $I_A = I_B = I_C$. It is obvious that L_A is a de Morgan implicative fuzzy algebra.

The implication I satisfy the (NP), so {1} is a proper filter of L_A, more precisely,
F = {1}is the unique proper filter of L_A is.

REFERENCES

[1] J. Fodor, M. Roubens (1994) Fuzzy Preference Modeling and Multi criteria Decision Support, Kluwer Academic Publishers, Dordrecht.

[2] E.P.Klement, R.Mesiar (2000) E.Pap, Triangular Norms, Kluwer, Dordrecht.

[3] B.Schweizer, A.Sklar (1983) Probabilistic Metric Spaces, North-Holland, Amsterdam.

[4] E.Trillas, L.Valverde, On implication and indistinguishability in the setting of fuzzy logic, in :J.Kacprzyk, R.R.Yager(Eds.), Management

[5] S.Gottwald,A (2001)Treatise on Many-valued Logic, Research Studies Press, Baldock.

[6] Baczynski, M., Jayaram, B. (2010). QL-implications: Some properties and intersections. Fuzzy Sets and Systems, 161(2), 158-188.

[7] Di Nola, A., and Ventre, A. G. (1989). On fuzzy implication in De Morgan algebras. Fuzzy sets and systems, 33(2), 155-164.

[8] M.Baczynski, B.Jayaram (2007), (S,N)-and R-implications : A state-of-the-art survey, Fuzzy Sets and systems. 159 1836-1859.

[9] Bedregal, B., Mezzomo, I., and Reiser, R. H. S. (2018) n -Dimensional Fuzzy Negations. IEEE Transactions on Fuzzy Systems.

[10] C.V. Hegoita (1998) Expert Systems and Fuzzy Systems (Benjamin/Cummings, Menlo Park, CA).

[11] Y. Xu and K. Y. Qin, On Filters of Lattice Implication Algebras, J. Fuzzy Math,, Vol. 1 No. 2 pp.251-260, Jun 1993.

[12] Y. B. Jun, Y. Xu and K. Y. (Jan 1998) Qin Positive Implicative Associative Filters of Lattice Implication Algebras, Bu11.Korean Math.Soc., Vol.35, No.1pp.53-61.

[13] B. L. Meng, (Jun 1998) Prime Filters of Lattice of Implication Algebras, J. Northwest University, Vol. 28 No. 3 pp. 189-192.

[14] M. H Wu, (2008)Study on the Lattice Implication Algebras and Correlative Substructures. Chengdou FJ. Southwest Jiaotong Univ.

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