Enumeration of the Number of Spanning Trees of the Globe Network and Its Subdivision

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Abstract

The complexity of a graph can be determined using a network-theoretic method, which is given. This strategy is based on the connection between graph theory and the theory of determinants in linear algebra. In this article, we provide a unique algebraic method that makes use of linear algebra to derive simple formulas for the complexity of diverse novel networks. We can derive the explicit formulas for the globe network and its subdivision using this method. In the final least, we also determine their spanning trees entropy and compare it to each other.

Mathematics Subject Classification: Primary 05C05, Secondary 05C30

Keywords spanning trees, Laplacian matrix, entropy

1 Introduction

One of the well-researched topics in graph theory is determining the number of spanning trees in a graph. We deal with simple and finite graphs $G = (V, E)$, where V denotes the vertex set and E denotes the edge set. A graph T is called a tree if it has not circuits so there is exactly one path connecting each vertex to every other vertex in the tree. Spanning tree of a graph G is a tree that includes every vertex in the graph. The number of spanning trees in G, also called, the complexity of the graph G, indicated by $\tau(G)$[1], can be found in many applications.

The most important application areas are reliability of networks [2-4], recalling specific chemical isomers [5], and determining the number of Eulerian circuits in a graph [1]. In instance, counting spanning trees is an important phase in many computation, bounds, and approximation algorithms for network dependability [6]. In a network that can be represented by a graph, intercommunication between all nodes of the network necessitates the existence of a spanning tree; hence increasing the number of spanning trees is a method of increasing

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reliability.

A classic technique called the matrix tree theorem, also known as Kirchhoff"s matrix-tree theorem [7] which states that the number of non-identical spanning trees of a graph *G* is equal to any cofactor of its Laplacian matrix *L*=*D*-*A*, in which *D* is the degree matrix and *A* is the adjacency matrix of the graph *G*.

Using Laplacian eigenvalues is a another way to count this number. The following formula was created by Kelmans and Chelnoknov [8]:

$$\tau(G) = \frac{1}{P}\prod_{i=1}^{P-1}\lambda_i \qquad (1.1)$$

where $p=\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ are the eigenvalues of the Laplacian matrix *L* and *G* is a connected graph with *p* vertices.The deletion-contraction method is one common way to find the number of spanning tress $\tau(G)$. This method, which is reliable, enables counting the number of spanning trees in a multi graph *G*. The fact that is used in this method.

$$\tau(G)= \tau(G-e)+ \tau(G/e) \qquad (1.2)$$

where *G-e* denotes the graph obtained by deleting an arbitrary edge *e*, and *G*/*e* denotes the graph obtained by contracting an arbitrary edge *e* [1, 9]. For more results, see [10- 26].

In this paper we contribute a new algebraic method which is based on Dodgson and Chio's method. It has an advantage that calculate determinants of $n \times n$ (n = 3) matrix, by reducing determinants to 2 Order.

2.Dodgson and Chio's Condensation Method: Chio's condensation is a method for analyzing an $n\times n$ determinant in terms of $(n-1)\times(n-1)$ determinants[27]:

$$A=\begin{vmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \vdots & \vdots \ \ddots & \vdots \\ a_{n1} & a_{n2}\cdots & a_{nn} \end{vmatrix} = \frac{1}{a_{11}^{n-2}} \begin{Vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots \quad \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{Vmatrix} \qquad (2.1)$$

In order to compute determinants of size $n\times n$, Dodgson's condensation method first expresses them in terms of determinants of size $(n-1) \times (n-1)$ before expressing the latter in terms of determinants of size $(n-2) \times (n-2)$.

Another method, proposed by Armend [28] is based on Dodgson and Chio's method, but differs from it in that it resolves the problem by computing four unique determinants of $(n-1) \times (n-1)$ Order (which can be derived from determinants of $n\times n$ order; if the first row and the first column, the first row and the last column, the last row and the first column, the last row and the last column and elements that belong to only one of unique determinants are removed, we should refer to them unique elements and one determinant of $(n-2)\times(n-2)$ order which is formed from $n\times n$ order determinant with elements $a_{i,j}$ with $i, j \neq 1, n$, on condition that the determinant of $(n-2)\times(n-2)\neq 0$.

Theorem 2.1 [28] States that every determinant of $n\times n$ ($n > 2$) order can be reduced into a determinant of 2×2 order by computing 4 determinants of $(n-1) \times (n-1)$ order and one determinant of $(n-2) \times (n-2)$ order, on condition that $(n-2)\times(n-2)$ order determinants to be different from zero.

A method for computing the determinants of the $n\times n$ order using the following formula is currently being presented:

$$A=\begin{vmatrix} a_{11} & a_{12}\cdots & a_{1n} \\ a_{21} & a_{22}\cdots & a_{2n} \\ \vdots & \vdots\ \ddots & \vdots \\ a_{n1} & a_{n2}\cdots & a_{nn} \end{vmatrix} = \frac{1}{|B|}\begin{vmatrix} |C| & |D| \\ |E| & |F| \end{vmatrix},\ |B| \neq 0 \tag{2.2}$$

The $|B|$ is $(n\text{-}2)\times(n\text{-}2)$ order determinant which is the interior determinant of determinant $|A|$ while $|C|$, $|D|$, $|E|$ and $|F|$ are unique determinants of $(n\text{-}1)\times(n\text{-}1)$order, which can be formed from $n\times n$ order determinant.

3. Main results

Theorem 3.1: The number of spanning trees of globe graph GL_n is
$\tau(GL_n) =(k+1)*2^k$, $k \geq 2$, where k is the number of blocks.

Proof:

Let $n = k+3$ and $q = 2k+2$ are the number of vertices and edges of GL_n respectively.

$$\tau(GL_n) = \begin{vmatrix} n-2 & -1 & -1 & \cdots & -1 \\ -1 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 2 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & 2 \end{vmatrix}$$

According to Dodgson and Chio's method, we have

$$\tau(GL_n) = \begin{vmatrix} n-2 & -1 & -1 & \cdots & -1 \\ -1 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 2 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & 2 \end{vmatrix} = \frac{1}{|B|}\begin{vmatrix} |C| & |D| \\ |E| & |F| \end{vmatrix} = (k+1)*2^k.\ |B|\neq 0 \text{ such that:}$$

$$|B| = \begin{vmatrix} 2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2 \end{vmatrix} = 2^k,\ |C| = \begin{vmatrix} n-2 & -1 & -1\ \cdots & -1 \\ -1 & 2 & 0\ \cdots & 0 \\ -1 & 0 & 2\ \ddots & \vdots \\ \vdots & \vdots & \ddots\ \ddots & 0 \\ -1 & 0 & \cdots\ 0 & 2 \end{vmatrix} = 2^{k-1}*(k+2)$$

$$|D| = \begin{vmatrix} -1 & = 1 & \cdots & \cdots & -1 \\ 2 & 0 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 2 & 0 \end{vmatrix} = 2^k,\ |E| = \begin{vmatrix} -1 & 2\ 0 & \cdots & 0 \\ -1 & 0\ \ddots & \ddots & \vdots \\ \vdots & \vdots\ \ddots & \ddots & 0 \\ \vdots & \vdots\ \ddots & \ddots & 2 \\ -1 & 0\ \cdots & \cdots & 0 \end{vmatrix} = 2^k$$

$$|F| = \begin{vmatrix} 2 & 0 & \dots & \dots & 0 \\ 0 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 2 \end{vmatrix} = 2^{k+1}$$

Therefore we get

$$\tau(GL_n) = \begin{vmatrix} n-2 & -1 & -1 & \cdots & -1 \\ -1 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 2 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & 2 \end{vmatrix} = \frac{1}{|B|}\begin{vmatrix} |C| & |D| \\ |E| & |F| \end{vmatrix} = \frac{1}{2^k}\begin{vmatrix} 2^{k-1} * (k+2) & 2^k \\ 2^k & 2^{k+1} \end{vmatrix} = (k+1)*2^k,$$

$k \geq 2$

Theorem 3.2: The number of spanning trees of subdivision **of** globe graph $S(GL_n)$ is $\tau(S(GL_n)) = (k+1)*4^k$, $k \geq 2$, where k is the number of blocks.

Proof:

Let $p=3k+5$ and $q=4k+4$ are the number of vertices and edges of $(S(GL_n))$ respectively.

$$\tau(S(GL_n)) = \begin{vmatrix} n-2/3 & -1 & \cdots & -1 & 0 & \cdots & 0 \\ -1 & 2 & 0 & \cdots & -1 & \ddots & \vdots \\ \vdots & 0 & 2 & \ddots & \ddots & \ddots & 0 \\ -1 & \vdots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & \cdots & -1 & 0 & \cdots & 2 \end{vmatrix}$$

According to Dodgson and Chio's method, we have

$\tau(S(GL_n)) = \frac{1}{|B|}\begin{vmatrix} |C| & |D| \\ |E| & |F| \end{vmatrix} = (k+1)*4^k$. $|B| \neq 0$ such that:

$$|B| = \begin{vmatrix} 2 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \cdots & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & \ddots & \ddots & \ddots & 2 \end{vmatrix} = 2^k$$

$$|C|=\begin{vmatrix} n-2/3 & -1 & \cdots & -1 & 0 & \cdots & 0 \\ -1 & 2 & 0 & \cdots & \ddots & \ddots & \vdots \\ \vdots & 0 & 2 & \ddots & \ddots & \ddots & 0 \\ -1 & \vdots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 2 \end{vmatrix} = 2^{k-1} * (k+2)$$

$$|D|=\begin{vmatrix} -1 & 2 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ -1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ 0 & -1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \cdots & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 2 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \end{vmatrix} = 2^{k}$$

$$|E|=\begin{vmatrix} -1 & \cdots & \cdots & -1 & 0 & \cdots & 0 \\ 2 & 2 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & -1 & 0 & \cdots & 0 & 2 \end{vmatrix} = 2^{k}$$

$$|F|=\begin{vmatrix} 2 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 2 \end{vmatrix} = 2^{k+1}$$

$$\tau(S(GL_n))=\frac{1}{|B|}\begin{vmatrix} |C| & |D| \\ |E| & |F| \end{vmatrix}=(k+1)*4^{k},\ k\geq 2.$$

4. Spanning Tree Entropy: The entropy of spanning trees of a network or the asymptotic complexity is a quantitative measure of the number of spanning trees and it characterizes the network structure. We use this entropy to quantify the robustness of networks. The most robust network is the network that has the highest entropy. We can calculate its spanning tree entropy which is a finite number and a very interesting quantity characterizing the network structure, defined as in [29] as

$$Z(G)=\lim_{V(G)\to\infty}\frac{\mathrm{In}\,\tau(G)}{|V(G)|} \longrightarrow \qquad (4.1)$$

Corollary 4.1: The entropy of spanning trees of the globe graph GL_n is
$Z(GL_n)=2$

Proof: From the Theorem 3. 1 and equation (4.1) and $|V(GL_n)| = n = k+3$ we obtain:
$\mathbf{Z(GL_n)}=\lim_{n\to\infty}\frac{\boldsymbol{In4^{n-3}}}{\boldsymbol{n}}=\lim_{n\to\infty}\frac{\boldsymbol{(n-3)In\,4}}{\boldsymbol{n}}=In\sqrt{4}=2.$
Corollary 4.2: The entropy of spanning trees of the subdivision of globe graph $S(GL_n)$ is
$Z(S(GL_n))=2$

Proof: From the Theorem 3. 2 and equation (a) and $|V(GL_n)| = n = 3k+5$ we obtain:

$$\mathbf{Z(GL_n)} = \lim_{n\to\infty} \frac{In 8^{\frac{n-5}{3}}}{n} = \lim_{n\to\infty} \frac{(n-5) In\, (8)^{\frac{1}{3}}}{n} = In\sqrt[3]{8} = 2.$$

5. Conclusion

In this paper, we contributed a new algebraic method to derive simple formulas for the complexity of some new networks using linear algebra. We applied this method to derive the explicit formulas for the globe network and its subdivision.

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