FITTED OPERATOR FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED PARABOLIC CONVECTION-DIFFUSION TYPE

Tesfaye Aga Bullo and Gemechis File Duressa

Department of Mathematics, College of Natural Science, Jimma University, Jimma, P.O. Box 378, Ethiopia

ABSTRACT

In this paper, we study the numerical solution of singularly perturbed parabolic convection-diffusion type with boundary layers at the right side. To solve this problem, the backward-Euler with Richardson extrapolation method is applied on the time direction and the fitted operator finite difference method on the spatial direction is used, on the uniform grids. The stability and consistency of the method were established very well to guarantee the convergence of the method. Numerical experimentation is carried out on model examples, and the results are presented both in tables and graphs. Further, the present method gives a more accurate solution than some existing methods reported in the literature.

KEYWORDS

Singularly perturbation; convection-diffusion; boundary layer; Richardson extrapolation; fitted operator.

1. INTRODUCTION

Singularly perturbed parabolic partial differential equations are types of the differential equations whose highest order derivative multiplied by small positive parameter. Basic application of these equations are in the Navier stoke's equations, in modeling and analysis of heat and mass transfer process when the thermal conductivity and diffusion coefficients are small and the rate of reaction is large [6, 11]. The presence of a small parameter in the given differential equation leads to difficulty to obtain satisfactory numerical solutions. Thus, numerical treatment of the singularly perturbed parabolic initial-boundary value problem is problematic because of the presence of boundary layers in its solution. Hence, time-dependent convection-diffusion problems have been largely discovered by researchers. Wherever they either use time discretization followed by the spatial discretization or discretizes all the variables at once.

Researchers like, Clavero et al. [2] used the classical implicit Euler to discretize the time variable and the upwind scheme on a non-uniform mesh for the spatial variable. The scheme was shown to be first-order accurate. To get a higher-order method, Kadalbajoo and Awasthi [9] used the Crank Nicolson finite difference method to discretize the time variable along with the classical finite difference method on a piecewise uniform mesh for the space variable. Gowrisankar and Natesan [13] working the backward Euler to discretize the time variable and the classical upwind finite difference method on a layer adapted mesh for the space variable. To hypothesis the mesh, the authors used equidistribution of a positive monitor function which is a linear combination of a constant and the second-order spatial derivative of the singular component of the solution at each time level. Their analysis provided a first-order accuracy in time and almost first-order accuracy in space. Again, Growsiankor and Natesan [12] again constructed a layer adapted mesh together with the classical upwind finite difference method to discretize the spatial variable.

In [9], Kadalbajoo et al. used the B-spline collocation method on a piecewise uniform mesh to discretize the spatial variable and the classical implicit Euler for time variable to obtain a higher-order accuracy in space. Natesan and Mukherjee [8] working using the Richardson extrapolation method to post-process a uniformly convergent method to a second-order accuracy in both variables. But, these methods mostly treated the stated problem for small order of convergence. Thus, it is necessary to improve the accuracy with higher order of convergence for solving singularly perturbed parabolic convection-diffusion types.

2. FORMULATION OF NUMERICAL METHOD

Consider the singularly perturbed parabolic initial-boundary value problems of the form

$$\frac{\partial u}{\partial t}(x,t)-\varepsilon\frac{\partial^2 u}{\partial x^2}(x,t)+a(x,t)\frac{\partial u}{\partial x}(x,t)+b(x,t)u(x,t)=f(x,t) \tag{1}$$

$$(x,t)\in\Omega:=(0,1)\times(0,T]$$

subject to the initial and boundary conditions:

$$\begin{aligned} u(x,0)&=s(x) \quad \text{on } S_x:=\{(x,0):0\le x\le 1\}\\ u(0,t)&=q_0(t) \quad \text{on } S_0:=\{(0,t):0<t\le T\}\\ u(1,t)&=q_1(t), \quad \text{on } S_1:=\{(1,t):0<t\le T\} \end{aligned} \tag{2}$$

where ε is a perturbation parameter that satisfy $0<\varepsilon<<1$. Functions $a(x,t)$, $b(x,t)$ and $f(x,t)$ are assumed to be sufficiently smooth functions on the given domain $\bar{\Omega}$ such that for the constants α and β.

$$\begin{aligned} a(x,t)&\ge\alpha>0,\\ b(x,t)&\ge\beta\ge 0. \end{aligned} \tag{3}$$

Under sufficient smoothness and compatibility conditions imposed on the functions $s(x),\ q_0(t),\ q_1(t)$ and $f(x,t)$, the initial-boundary value problem admits a unique solution $u(x,t)$ to the assumed condition in Eq. (3), $a(x,t)>0$ which exhibits a boundary layer of width $O(\varepsilon)$ near the boundary $x=1$ of the domain Ω, [8].

Consider Eq. (1) on a particular domain $(x,t)\in\Omega:=(0,1)\times(0,1]$ with the conditions in Eq. (2) and let M and N be positive integers. When working on $\bar{\Omega}$, we use a rectangular grid Ω_h^k whose nodes are (x_m,t_n) for $m=0,1,\ldots M$ and $n=0,1,\ldots N$. Here, $0=x_0<x_1<\ldots<x_M=1$ and $0=t_0<t_1<\ldots<t_N=1$ such grids are called tensor-product grids.

$$t_n = nk,\ k = \frac{1}{N},\ n = 0,1,2,\ldots,N \text{ and } x_m = mh,\ h = \frac{1}{M},\ m = 0,1,2,\ldots,M \tag{4}$$

Let denote the approximate solution $u_m^n \approx u(x_m, t_n)$ at an arbitrary point (x_m, t_n). Then, considering the uniform discretization formula given in Eq. (4) in the $t-$direction only, we have the finite difference approximation for the derivatives with respect to *t* as:

$$\frac{\partial u^{n+1}}{\partial t}(x) = \frac{u^{n+1}(x) - u^n(x)}{k} - \frac{k}{2}\frac{\partial^2 u^{n+1}}{\partial t^2}(x) + \ldots \tag{5}$$

Substituting Eq. (5) into Eq. (1) transforms the singularly perturbed partial differential equation to ordinary differential equation at each $(n+1)^{th} -$ level of the form:

$$-\varepsilon\frac{d^2u^{n+1}}{dx^2}(x) + a^{n+1}(x)\frac{du^{n+1}}{dx}(x) + \left(\frac{1}{k} + b^{n+1}(x)\right)u^{n+1}(x) = f^{n+1}(x) + \frac{u^n(x)}{k} - \tau_1 \tag{6}$$

where $\tau_1 = -\frac{k}{2}\frac{\partial^2 u^{n+1}}{\partial t^2}(x)$. With the conditions: $u(0,t) = q_0(t)$ and $u(1,t) = q_1(t), \quad 0 < t \le 1$
Let denoting:

$$B^{n+1}(x) = a^{n+1}(x)\frac{du^{n+1}}{dx}(x) + \left(\frac{1}{k} + b^{n+1}(x)\right)u^{n+1}(x) - f^{n+1}(x) - \frac{u^n(x)}{k} + \tau_1 \tag{7}$$

Then, Eq. (6) re-written as the second order singularly perturbed ordinary boundary value problem

$$\varepsilon\frac{d^2u^{n+1}}{dx^2}(x) = B^{n+1}(x) \tag{8}$$

Subject to the boundary conditions: $u(0,t) = q_0(t) = u_0^{n+1}$ and $u(1,t) = q_1(t) = u_M^{n+1}, \quad 0 < t \le 1$.

A general $j-$step method for the solution of Eq. (8) can be written as:

$$\varepsilon u_{m+1}^{n+1} = \varepsilon\sum_{i=1}^{j} e_i u_{m-i+1}^{n+1} + h^2\sum_{i=0}^{j} g_i\left(\frac{d^2u}{dx^2}\right)_{m-i+1}^{n+1} \tag{9}$$

Where e_i's and g_i's are arbitrary constant.

To determine the coefficients e_i's and g_i's, write the local truncation error (LT_{m+1}^{n+1}) for $j = 2$, in expanding form as:

$$LT_{m+1}^{n+1} = \varepsilon u^{n+1}(x_{m+1}) - \varepsilon e_1 u^{n+1}(x_m) - \varepsilon e_2 u^{n+1}(x_{m-1}) - h^2\left(g_0\left(\frac{d^2u}{dx^2}(x_{m+1})\right)^{n+1} + g_1\left(\frac{d^2u}{dx^2}(x_m)\right)^{n+1} + g_2\left(\frac{d^2u}{dx^2}(x_{m-1})\right)^{n+1}\right) \quad (10)$$

Assume that the function $u^{n+1}(x)$ has continuous derivatives of sufficiently higher order. Expanding the terms $u^{n+1}(x_{m\pm1})$ and $\left(\frac{d^2u}{dx^2}(x_{m\pm1})\right)^{n+1}$ by Taylor's series expansion about the point x_m then substituting into Eq. (11) and grouping like terms gives:

$$\begin{aligned}LT_{m+1}^{n+1} &= \varepsilon\left(1-e_1-e_2\right)u^{n+1}(x_m) + \varepsilon h(1+e_2)\left(\frac{du}{dx}(x_m)\right)^{n+1} + \\ &h^2\left(\frac{\varepsilon}{2} - \varepsilon e_2 - g_0 - g_1 - g_2\right)\left(\frac{d^2u}{dx^2}(x_m)\right)^{n+1} + \\ &h^3\left(\frac{\varepsilon}{6} + \frac{\varepsilon e_2}{6} - g_0 + g_2\right)\left(\frac{d^3u}{dx^3}(x_m)\right)^{n+1} + h^4\left(\frac{\varepsilon}{24} - \frac{\varepsilon e_2}{24} - \frac{g_0}{2} - \frac{g_2}{2}\right)\left(\frac{d^4u}{dx^4}(x_m)\right)^{n+1} + \\ &h^5\left(\frac{\varepsilon}{120} + \frac{\varepsilon e_2}{120} - \frac{g_0}{6} + \frac{g_2}{6}\right)\left(\frac{d^5u}{dx^5}(x_m)\right)^{n+1} + h^6\left(\frac{\varepsilon}{720} - \frac{\varepsilon e_2}{720} - \frac{g_0}{24} - \frac{g_2}{24}\right)\left(\frac{d^6u}{dx^6}(x_m)\right)^{n+1} + \ldots\end{aligned} \quad (11)$$

Method given in Eq. (9), for $j = 2$ is of order four if all the coefficients given in Eq. (11) are equal to zero except after the coefficient of h^6, which gives:

$$e_1 = 2,\ e_2 = -1,\ g_0 = g_2 = \frac{1}{12} \text{ and } g_1 = \frac{5}{6}$$

Because of $j = 2$ and the values of this system of equations, Eq. (9) becomes:

$$\varepsilon\left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right) = \frac{h^2}{12}\left(\left(\frac{d^2u}{dx^2}\right)_{m-1}^{n+1} + 10\left(\frac{d^2u}{dx^2}\right)_m^{n+1} + \left(\frac{d^2u}{dx^2}\right)_{m+1}^{n+1}\right) + \tau_2 \quad (12)$$

Where $\tau_2 = -\frac{h^4}{240}\left(\frac{d^6u}{dx^6}\right)_m^{n+1}$

Using the denotation in Eq. (7) into Eq. (13), we have:

$$\frac{12\varepsilon}{h^2}\left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right) = B_{m-1}^{n+1} + 10B_m^{n+1} + B_{m+1}^{n+1} + \tau_2 \quad (13)$$

Consider the following Taylor's series expansion about the point x_m as:

$$\frac{du_{m+1}^{n+1}}{dx}=\frac{du_{m}^{n+1}}{dx}+h\frac{d^2u_{m}^{n+1}}{dx^2}+O(h^2) \quad \text{and} \quad \frac{du_{m-1}^{n+1}}{dx}=\frac{du_{m}^{n+1}}{dx}-h\frac{d^2u_{m}^{n+1}}{dx^2}+O(h^2) \tag{14}$$

With the central difference approximation:

$$\frac{du_{m}^{n+1}}{dx}=\frac{u_{m+1}^{n+1}-u_{m-1}^{n+1}}{2h}+\tau_3 \quad \text{and} \quad \frac{d^2u_{m}^{n+1}}{dx^2}=\frac{u_{m-1}^{n+1}-2u_{m}^{n+1}+u_{m+1}^{n+1}}{h^2}+\tau_4 \tag{15}$$

where $\tau_3=-\frac{h^2}{6}(\frac{d^3u}{dx^3})_m^{n+1}$ and $\tau_4=-\frac{h^2}{12}(\frac{d^4u}{dx^4})_m^{n+1}$

Substituting Eq. (15) into Eq. (14), also gives the central difference approximation:

$$\frac{du_{m+1}^{n+1}}{dx}=\frac{3u_{m+1}^{n+1}-4u_{m}^{n+1}+u_{m-1}^{n+1}}{2h}+\tau_5 \quad \text{and} \quad \frac{du_{m-1}^{n+1}}{dx}=\frac{-u_{m+1}^{n+1}+4u_{m}^{n+1}-3u_{m-1}^{n+1}}{2h}+\tau_6 \tag{16}$$

Where $\tau_5=\tau_3+h\tau_4+O(h^2)$ and $\tau_6=\tau_3-h\tau_4+O(h^2)$

Now, considering Eqs. (15) and (16) into Eq. (7) at $x=x_m$ and $x=x_{m\pm1}$ gives:

$$\begin{aligned}
B_{m-1}^{n+1}&=\frac{a_{m-1}^{n+1}}{2h}(-u_{m+1}^{n+1}+4u_{m}^{n+1}-3u_{m-1}^{n+1})+\tau_3a_{m-1}^{n+1}+\left(\frac{1}{k}+b_{m-1}^{n+1}\right)u_{m-1}^{n+1}-f_{m-1}^{n+1}-\frac{1}{k}u_{m-1}^{n}+\tau_1\\
B_{m}^{n+1}&=\frac{a_{m}^{n+1}}{2h}(u_{m+1}^{n+1}-u_{m-1}^{n+1})+\tau_3a_{m}^{n+1}+\left(\frac{1}{k}+b_{m}^{n+1}\right)u_{m}^{n+1}-f_{m}^{n+1}-\frac{1}{k}u_{m}^{n}+\tau_1\\
B_{m+1}^{n+1}&=\frac{a_{m+1}^{n+1}}{2h}(3u_{m+1}^{n+1}-4u_{m}^{n+1}+u_{m-1}^{n+1})+\tau_3a_{m+1}^{n+1}+\left(\frac{1}{k}+b_{m+1}^{n+1}\right)u_{m+1}^{n+1}-f_{m+1}^{n+1}-\frac{1}{k}u_{m+1}^{n}+\tau_1
\end{aligned} \tag{17}$$

Substituting Eq. (17) into Eq. (13) and grouping like terms, we get:

$$\begin{aligned}
&-\left(\frac{12\varepsilon}{h^2}+\frac{3a_{m-1}^{n+1}}{2h}-\frac{1}{k}-b_{m-1}^{n+1}+\frac{5a_{m}^{n+1}}{h}-\frac{a_{m+1}^{n+1}}{2h}\right)u_{m-1}^{n+1}\\
&+\left(\frac{24\varepsilon}{h^2}+\frac{2a_{m-1}^{n+1}}{h}+10\left(\frac{1}{k}+b_{m}^{n+1}\right)-\frac{2a_{m+1}^{n+1}}{h}\right)u_{m}^{n+1}\\
&-\left(\frac{12\varepsilon}{h^2}+\frac{a_{m-1}^{n+1}}{2h}-\frac{1}{k}-b_{m+1}^{n+1}+\frac{5a_{m}^{n+1}}{h}-\frac{3a_{m-1}^{n+1}}{2h}\right)u_{m+1}^{n+1}=\frac{1}{k}\left(u_{m-1}^{n}+10u_{m}^{n}+u_{m+1}^{n}\right)+\\
&\qquad f_{m-1}^{n+1}+10f_{m}^{n+1}+f_{m+1}^{n+1}+T_{m}^{n+1}
\end{aligned} \tag{18}$$

Where the truncation error T_m^{n+1} is given by:

$$T_m^{n+1}=12\tau_1+\tau_2+\tau_3a_m^{n+1}+\tau_5a_{m+1}^{n+1}+\tau_6a_{m-1}^{n+1}$$

Introducing fitting parameter σ into Eq. (18), let denote $\rho = \frac{h}{\varepsilon}$ and then evaluate the limit of Eq. (18) as $h \to 0$ we get:

$$\sigma = \frac{\rho a_{(1)}^{n+1} \lim_{h\to 0}\left(u_{m+1}^{n+1} - u_{m-1}^{n+1}\right)}{2\lim_{h\to 0}\left(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}\right)} \tag{19}$$

On the other hand, as the full prove given by Roos *et. al.*, [6], the asymptotic expansion for the solution Eq. (1) is given:

$$u^{n+1}(x) = u_0^{n+1}(x) + Ae^{-a_{(1)}^{n+1}(\frac{1-x}{\varepsilon})} \tag{20}$$

where $u_0^{n+1}(x)$ is the solution of the reduced problem, and A is an arbitrary constant. From Eq. (20), we have:

$$\begin{aligned}
\lim_{h\to 0} u_m^{n+1} &= u_0^{n+1}(0) + Ae^{-\frac{a^{n+1}(1)}{\varepsilon}} e^{a^{n+1}(1)(m\rho)} \\
\lim_{h\to 0} u_{m\pm 1}^{n+1} &= u_0^{n+1}(0) + Ae^{-\frac{a^{n+1}(1)}{\varepsilon}} e^{a^{n+1}(1)((m\pm 1)\rho)}
\end{aligned} \tag{21}$$

Putting Eq. (21) into Eq. (19), gives the value of fitting parameter σ as:

$$\sigma = \frac{\rho a_{(1)}^{n+1}}{2} \coth\left(\frac{\rho a_{(1)}^{n+1}}{2}\right) \tag{22}$$

The obtained scheme, after introducing the fitted parameter σ which can be written as the three-term recurrence relation:

$$E_m^{n+1} u_{m-1}^{n+1} + F_m^{n+1} u_m^{n+1} + G_m^{n+1} u_{m+1}^{n+1} \quad = H_m^{n+1} + TE_m^{n+1} \tag{23}$$

For $m = 1, 2, \ldots, M; \quad n = 0, 1, \ldots, N$ and $\lambda = \frac{k}{h}$

Where $E_m^{n+1} = -\lambda\left(\frac{12\varepsilon\sigma}{h} + \frac{1}{2}(3a_{m-1}^{n+1} - a_{m+1}^{n+1}) + 5a_m^{n+1}\right) + 1 + kb_{m-1}^{n+1}$

$$F_m^{n+1} = 2\lambda\left(\frac{12\varepsilon\sigma}{h} + a_{m-1}^{n+1} - a_{m+1}^{n+1}\right) + 10 + 10kb_m^{n+1}$$

$$G_m^{n+1} = -\lambda\left(\frac{12\varepsilon\sigma}{h} + \frac{1}{2}(a_{m+1}^{n+1} - 3a_{m+1}^{n+1}) - 5a_m^{n+1}\right) + 1 + kb_{m+1}^{n+1}$$

$$H_m^{n+1} = u_{m-1}^n + 10u_m^n + u_{m+1}^n + k\left(f_{m-1}^{n+1} + 10f_m^{n+1} + f_{m+1}^{n+1}\right)$$

With the truncation error $TE_m^{n+1} = kT_m^{n+1}$

Since the initial and boundary conditions are given in Eq. (2), we have $u(x,0) = s(x) = u_m^0$, $\forall m = 0,1, \ldots M$, $u(0,t) = q_0(t) = u_0^{n+1}$ and $u(1,t) = q_1(t) = u_M^{n+1}$ $\forall n = 0,1, \ldots N$

3. Richardson Extrapolation

Richardson extrapolation technique is a convergence acceleration technique that consists of two computed approximations of a solution (on two nested meshes) [16]. The linear combination turns out to be a better approximation. The truncation error of the schemes in Eq. (18) is given by:

$$T_m^{n+1} = 12\tau_1 + \tau_2 + \tau_3 a_m^{n+1} + \tau_5 a_{m+1}^{n+1} + \tau_6 a_{m-1}^{n+1} = -6k\frac{\partial^2 u}{\partial t^2}(x) - \frac{h^4}{240}\left(\frac{d^6 u}{dx^6}\right)_m^{n+1} = O\left(k + h^4\right)$$

Hence, we have

$$\left|u(x_m, t_n) - U_m^n\right| \le C\left(k + h^4\right) \tag{24}$$

Where $u(x_m, t_n)$ and U_m^n are exact and approximate solutions respectively, C is a constant independent of mesh sizes h and k.

Let Ω_M^{2N} be the mesh obtained by bisecting each mesh interval in Ω_M^N and denote the approximation of the solution on Ω_M^{2N} by $\bar{U}_m^n$. Consider Eq. (24) works for any $h, k \ne 0$, which implies:

$$u(x_m, t_n) - U_m^n \le C\left(k + h^4\right) + R_M^N, \quad (x_m, t_n) \in \Omega_M^N \tag{25}$$

So that, it works for any $h, \dfrac{k}{2} \ne 0$ yields:

$$u(x_m, t_n) - \bar{U}_m^n \le C\left(\frac{k}{2} + (h)^4\right) + R_M^{2N}, \quad (x_m, t_n) \in \Omega_M^{2N} \tag{26}$$

Where the remainders, R_M^N and R_M^{2N} are $O(k^2 + h^4)$. A combination of inequalities in Eqs. (25) and (26) leads to $u(x_m, t_n) - \left(2\bar{U}_m^n - U_m^n\right) = O(k^2 + h^4)$ which suggests that

$$\left(U_m^n\right)^{ext} = 2\bar{U}_m^n - U_m^n \tag{27}$$

is also an approximation of $u(x_m, t_n)$.

Using this approximation to evaluate the truncation error, we obtain

$$\left|u(x_m,t_n)-\left(U_m^n\right)^{ext}\right| \le C\left(k^2+h^4\right) \tag{28}$$

4. STABILITY AND CONVERGENCE ANALYSIS

A partial differential equation is well-posed if its solution exists, and depends continuously on the initial and boundary conditions as given in [5 - 6]. The Von Neumann stability technique is applied to investigate the stability of the developed scheme in Eq. (23), by assuming that the solution of Eq. (23) at the grid point $\left(x_m,t_n\right)$ is given by:

$$u_m^n = \xi^n e^{i\,m\theta} \tag{29}$$

where $i=\sqrt{-1},$ and θ is the real number and ξ is the amplitude factor. Now, putting Eq. (29) into the homogeneous part of Eq. (23) gives:

$$\xi = \frac{e^{-i\theta}+10+e^{i\theta}}{E_m^{n+1}e^{-i\theta}+F_m^{n+1}+G_m^{n+1}e^{i\theta}}$$

Since, the value of $i=\sqrt{-1}$ and using the relations: $e^{\pm i\theta}=\cos\theta \pm i\sin\theta$, the above equation becomes:

$$\xi = \frac{10+2\cos\theta}{F_m^{n+1}+\left(E_m^{n+1}+G_m^{n+1}\right)\cos\theta+i\left(E_m^{n+1}-G_m^{n+1}\right)\sin\theta} \tag{30}$$

From the trigonometric definition, the value $\max_{\forall\theta}\cos|\theta|=1$ which simplifies the relation and from Eq. (3), we have: $a(x,t)\ge\alpha>0$ and $b(x,t)\ge\beta\ge0$. Thus, Eq. (30), show that the generalization for stability:

$$|\xi|^2 = \frac{124}{144+\ 4\lambda k\left(\dfrac{12\varepsilon\sigma}{h}\right)+24\lambda a_m^{n+1}+2\dfrac{12\lambda\varepsilon\sigma}{h}+4\left(6\lambda a_m^{m+1}\right)^2} \le \frac{124}{144} < 1$$

Therefore, $\xi\ <1$

Hence, the developed scheme in Eq. (23) is stable. Thus, the developed scheme in Eq. (23) is unconditionally stable by Lax Richtmyer's definition, [5].

To investigate the consistency of the method, we have the truncation error from Eq. (23) given as:- $TE_m^{n+1}=kT_m^{n+1},$ where $T_m^{n+1}=12\tau_1+\tau_2+\tau_3 a_m^{n+1}+\tau_5 a_{m+1}^{n+1}+\tau_6 a_{m-1}^{n+1}$, which re-arranged as:

$$TE_m^{n+1} = -6k\frac{\partial^2 u}{\partial t^2}(x)+O(h^4) \tag{31}$$

Thus, Eq. (31) vanishes as $k \to 0$ and $h \to 0$. Hence, the scheme is consistent with the order of $O\left(k + h^2\right)$. Therefore, the scheme in Eq. (23), is convergent by Lax's equivalence theorem [5].

5. NUMERICAL EXAMPLES, RESULTS, AND DISCUSSION

Example 1: Consider the singularly perturbed parabolic initial-boundary value problem:

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + (1 + x(1-x)) \frac{\partial u}{\partial x} = f(x,t), \quad (x,t) \in (0,1) \times (0,1]$$

$$\begin{cases} u(x,0) = s(x), & 0 < x < 1 \\ u(0,t) = u(1,t) = 0, & 0 \le t \le 1 \end{cases}$$

We choose the initial data $s(x)$ and the source function $f(x,t)$ to fit with the exact solution:

$$u(x,t) = e^{-t}\left(e^{-\frac{1}{\varepsilon}} + x\left(1 - e^{-\frac{1}{\varepsilon}}\right) - e^{-\frac{1-x}{\varepsilon}}\right)$$

As the exact solution for this example is known, for each ε, we calculate absolute maximum error by: $E_\varepsilon^{M,N} = \max_{(x_i,t_j) \in Q_\varepsilon^{M,N}} \left| u(x_m,t_n) - \left(u_m^n\right)^{ext} \right|$, where $u(x_m,t_n)$ and $\left(u_m^n\right)^{ext}$ respectively, denote the exact and numerical solution. Besides, we determined the corresponding order of convergence by: $P_\varepsilon^{M,N} = \dfrac{\log(E_\varepsilon^{M,N}) - \log(E_\varepsilon^{2M,2N})}{\log(2)}$

Table 1: Comparison of Maximum absolute errors of the solution for Example 1 at the number of intervals M/N

ε	$32/10$	$64/20$	$128/40$	$256/80$	$512/160$	$1024/320$
Present method						
10^0	2.2301e-05	7.0395e-06	2.0147e-06	5.4221e-07	1.4096e-07	3.5958e-08
10^{-2}	6.3211e-03	1.5103e-03	2.9093e-04	6.7722e-05	1.6736e-05	4.1635e-06
10^{-4}	8.9601e-03	4.7439e-03	2.4923e-03	1.2798e-03	6.4877e-04	3.2620e-04
Meth0d in [3]						
10^0	6.8921e-04	3.7085e-04	1.9290e-04	9.8440e-05	4.9739e-05	---
10^{-2}	7.1532e-02	4.5000e-02	2.6393e-02	1.4579e-02	7.1423e-03	---
10^{-4}	9.3382e-02	5.5430e-02	3.9185e-02	2.1997e-02	1.1787e-02	---
Method in [12]						
10^0	6.8921e-04	3.7085e-04	1.9290e-04	9.8440e-05	4.9739e-05	2.5002e-05
10^{-2}	3.6778e-02	2.2324e-02	1.2953e-02	7.2482e-03	3.9080e-03	2.0439e-03
10^{-4}	6.6889e-02	3.7859e-02	2.0136e-02	1.0334e-02	5.2851e-03	2.6686e-03

Table 2: Comparisons of the corresponding order of convergence for Example 1 at the number of intervals M / N

ε	$32/10$	$64/20$	$128/40$	$256/80$	$512/160$
Present method					
10^0	1.6636	1.8049	1.8936	1.9436	1.9709
10^{-2}	2.0653	2.3761	2.1030	2.0167	2.0071
10^{-4}	0.9174	0.9286	0.9616	0.9801	0.9920
Meth0d in [3]					
10^0	0.8941	0.9430	0.9705	0.9849	---
10^{-2}	0.6687	0.7698	0.8563	1.0294	---
10^{-4}	0.7525	0.5004	0.8330	0.9001	---
Meth0d in [12]					
10^0	0.8941	0.9430	0.9705	0.9849	0.9923
10^{-2}	0.7203	0.7853	0.8376	0.8912	0.9351
10^{-4}	0.8211	0.9108	0.9624	0.9674	0.9858

Table 3: Maximum absolute errors and order of convergence before and after extrapolation for Example 1 at number of intervals $M = N$

ε	Extrapolation	$M = 32$	$M = 64$	$M = 128$	$M = 256$	$M = 512$
	Before	6.1530e-03	2.1777e-03	1.0045e-03	4.8793e-04	2.4106e-04
2^{-6}		1.4985	1.1163	1.0417	1.0173	---
	After	3.8402e-03	7.1539e-04	1.6307e-04	3.9538e-05	9.8539e-06
		2.4244	2.1332	2.0442	2.0045	---
	Before	1.0587e-02	4.7844e-03	1.6105e-03	5.7997e-04	2.7077e-04
2^{-8}		1.1459	1.5708	1.4735	1.0989	----
	After	8.8146e-03	3.9827e-03	1.0913e-03	2.0090e-04	4.6361e-05
		1.1461	1.8677	2.4415	2.1155	---
	Before	1.0639e-02	5.4361e-03	2.7310e-03	1.2154e-03	4.0731e-04
2^{-10}		0.9687	0.9931	1.1680	1.5772	---
	After	8.8727e-03	4.7569e-03	2.4582e-03	1.0617e-03	2.8578e-04
		0.8994	0.9524	1.2112	1.8934	--

Table 4: Maximum absolute errors with and without fitting factor for Example1 at number of intervals $M = N$

ε	32	64	128	256	512
With fitting factor					
2^{-6}	3.8402e-03	7.1539e-04	1.6307e-04	3.9538e-05	9.8539e-06
2^{-10}	8.8727e-03	4.7569e-03	2.4582e-03	1.0617e-03	2.8578e-04
2^{-14}	8.8727e-03	4.7570e-03	2.4744e-03	1.2659e-03	6.5539e-04
Without fitting factor					
2^{-6}	1.1678e-01	3.0846e-02	7.2058e-03	1.8335e-03	4.9938e-04
2^{-10}	7.7390e-01	7.6766e-01	5.7619e-01	3.4266e-01	1.3274e-01
2^{-14}	1.4193e+00	9.7878e-01	9.7093e-01	9.6259e-01	8.9013e-01

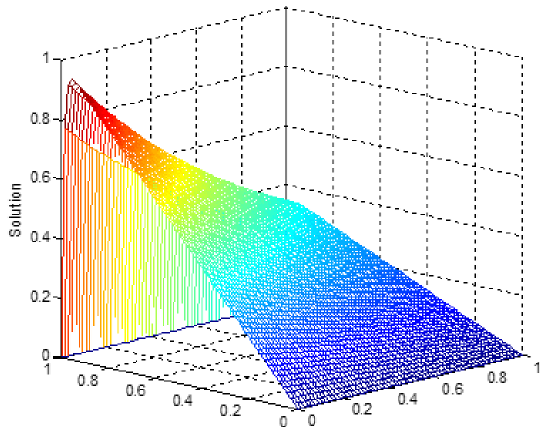

Figure 1: The physical behavior of the solutions of Example 1 for $M = N = 64$ and $\varepsilon = 10^{-2}$

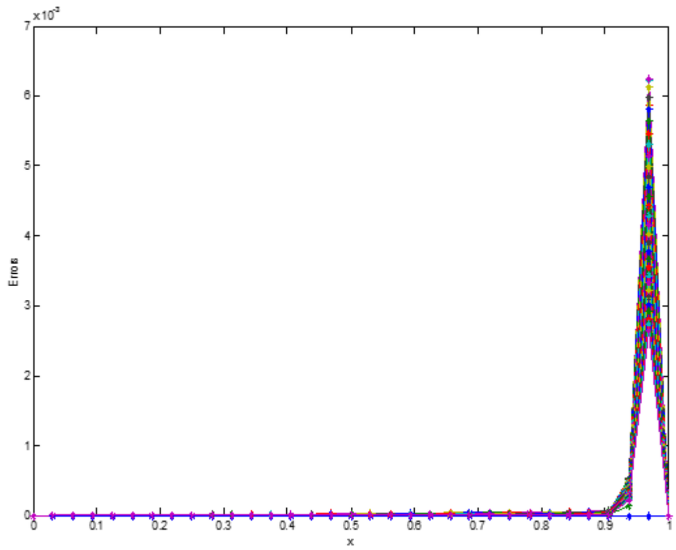

Figure 2: Pointwise absolute errors of Example 1 for $M = N = 32$ and $\varepsilon = 10^{-2}$

Example 2: Consider the following time-dependent convection-diffusion problem:

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (1 + x^2 + \frac{1}{2}\sin(\pi x))\frac{\partial u}{\partial x} + (1 + x^2 + \frac{1}{2}\sin(\frac{\pi t}{2}))u = x^3(1-x)^3 + t(1-t)\sin(\pi t)$$
$$(x,t) \in (0,1)\times(0,1]$$
$$u(x,0) = 0, \qquad 0 < x < 1$$
$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le 1$$

As the exact solution $u(x,t)$ is unknown, we use the double mesh principle

Table 5: Comparison of maximum absolute errors obtained by present method for Example 2 at the number of intervals M / N

ε	$32/16$	$64/32$	$128/64$	$256/128$
With fitting factor				
10^{0}	3.8982e-05	1.0566e-05	2.7520e-06	7.0238e-07
10^{-1}	3.1182e-04	8.6811e-05	2.2884e-05	5.8797e-06
10^{-2}	1.6457e-03	6.4773e-04	1.9137e-04	5.0168e-05
10^{-3}	1.7926e-03	9.8907e-04	5.1915e-04	2.5586e-04
10^{-4}	1.7926e-03	9.8907e-04	5.1957e-04	2.6634e-04
Without fitting factor				
10^{0}	4.0628e-05	1.0975e-05	2.8544e-06	7.2796e-07
10^{-1}	5.1449e-04	1.3026e-04	3.2566e-05	8.1845e-06
10^{-2}	3.6028e-02	1.5329e-02	4.5671e-03	9.9486e-04
10^{-3}	1.1039e-01	9.7292e-02	7.4558e-02	4.6373e-02
10^{-4}	1.2702e-01	1.2773e-01	1.2467e-01	1.1760e-01

Table 6: Comparisons of the corresponding order of convergence for Example 2 at M / N

ε	$32/16$	$64/32$	$128/64$
With fitting factor			
10^{0}	1.8834	1.9409	1.9702
10^{-1}	1.8448	1.9235	1.9605
10^{-2}	1.3452	1.7590	1.9315
10^{-3}	0.8579	0.9299	1.0208
10^{-4}	0.8579	0.9288	0.9640
Without fitting factor			
10^{0}	1.8883	1.9430	1.9713
10^{-1}	1.9817	2.0000	1.9924
10^{-2}	1.2329	1.7469	2.1987
10^{-3}	0.1822	0.3840	0.6851
10^{-4}	-0.0080	0.0350	0.0842

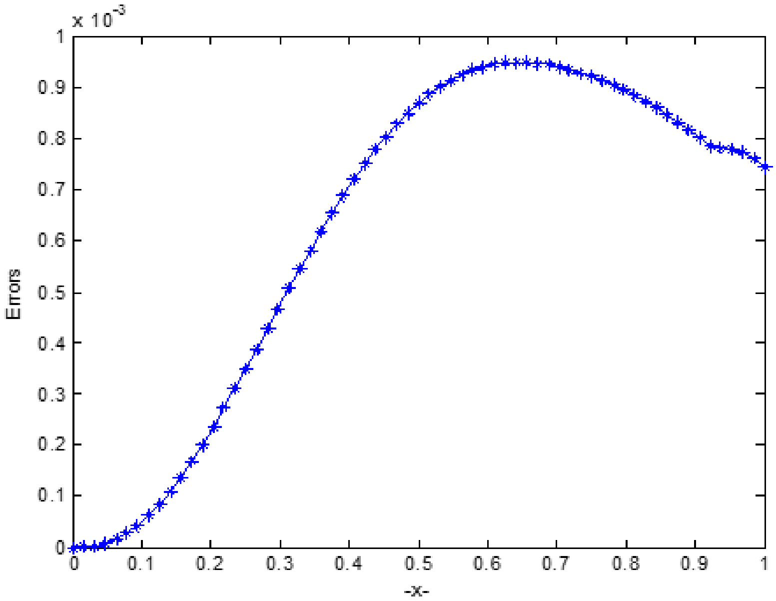

Figure 3: Pointwise absolute errors of Example 2 for $M = N = 64$ and $\varepsilon = 10^{-4}$

It can be seen from the results obtained and presented in Tables 1, 3, 4, and 5 show that the numerical method presented in this paper produces more accurate numerical solutions for singularly perturbed parabolic convection-diffusion problems with the right boundary layer. Results presented in Tables 2, 3, and 6, shows that the maximum absolute errors and the corresponding rate of convergence calculated using the present method is more accurate with a higher rate of convergence than some existing methods. For each ε, M and N in Tables 2 - 6 shows the effectiveness of applying the fitted operator finite difference method with Richardson extrapolation to obtain a more accurate numerical solution. The three figures (Figure 1 -3), show that the stated problem has the right boundary layer so that maximum absolute errors existing in the layer region.

6. CONCLUSION

In this paper, we have discussed a fitted operator finite difference method with Richardson extrapolation for solving singularly perturbed parabolic convection-diffusion problem. First, the Backward-Euler method is applied concerning the time derivative which gives a second-order singularly perturbed ordinary differential equation with two-point boundary value points. Second, applying the fitted operator finite difference method on the obtained ordinary differential equation results in the two-level in the time direction and three-term recurrence relations in spatial derivatives that can be solved by the Thomas algorithm. In this work, the two main contributions to obtain a more accurate numerical solution with the higher order of convergence are using second-order Richardson extrapolation in the time direction and introduce the fitting parameter on the spatial direction.

Numerical simulations are provided to support the theoretical descriptions and to demonstrate the effectiveness and betterment of using the proposed method. We point out that the fitting operator method used in this work can be extended to other types of singularly perturbed time-dependent problems.

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