# SKEW CONSTACYCLIC CODES OVER THE RING $F_q[u; v]/ < u^2 – 1, v^2 -1, uv - vu >$

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***ABSTRACT***

Skew constacyclic codes are displaying vital role in the field of coding theory. This paper is mainly focused on skew –α constacyclic codes over $R = F_q + uF_q + vF_q + uvF_q$, with $u^2 = v^2 = 1$, $uv = vu$, $q = 3^m$, and α is an unit of R fixed by the authomorphism θ. It is briey derived as well that the Gray map of skew cyclic code of length n over R is skew quasicyclic code of length 4n over Fq of index 4. Furthermore, the decomposition of a skew α constacyclic codes over R are also determined. A complete analysis about the relationship between skew α constacyclic code and its dual is also carried out.

***KEYWORDS***

Gray map, skew constacyclic codes, finite ring, negacyclic codes, skew polynomial ring.

## 1. INTRODUCTION

Skew cyclic codes over finite ring has gotten much attention of the researchers since after the first idea of skew cyclic codes which is a natural generalization of cyclic codes on skew polynomial ring ( [ 1 ] - [ 8 ] ) . L.F.T . Cuitio et al . determined some basic

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properties of skew generalized cyclic (GC) codes over finite fields ([9]). Furthermore, T. Yao et al. also studied the decomposition of skew cyclic code over the finite chain ring $H = F_q + uF_q + vF_q + uvF_q$ with $u^2 = u$, $v^2 = v$, $uv = vu$, $q = p^m$, with $p$ odd prime ( according to Frobenius automorphism)([10]). Later on, H. Islam et al. introduced the Frobenius automorphism over $H$ ([11]). Similarly, A. Dertli et al. defined the Gray map from $\dfrac{F_q[u,v]}{(u^2-1, v^2-1, uv-vu)}$ to $F_q^4$, and determined the decomposition of skew cyclic code over the ring $\dfrac{F_q[u,v]}{(u^2-1, v^2-1, uv-vu)}$ ([12]).

In this article we have described the skew (a + a u + a v + a uv)-constacyclic 3 code over the ring $\frac{F_q[u,v]}{(u^2-1,v^2-1,uv-vu)}$, where $q = 3^m$, and is also proved that the Gray image of skew cyclic code of length $n$ over $\dfrac{F_q[u,v]}{(u^2-1, v^2-1, uv-vu)}$ is skew quasi-cyclic code over $F_q$ having length $4n$ and index 4. This paper is organized as follow: In section 2, we will give some properties about the ring $R = \dfrac{F_q[u,v]}{(u^2-1, v^2-1, uv-vu)}$, and recall some auxiliary results in coding theory. Section 3 define the Gray map, and describe the $F_q$-image of skew cyclic code over $R$, and characterization of skew constacyclic codes over $R$ is presented in section 4.

## 2 Preliminaries

Let $q = 3^m$ and $R = \dfrac{F_q[u,v]}{(u^2-1, v^2-1, uv-vu)}$ which is isomorphic to $F_q + uF_q + vF_q + uvF_q$, with $u^2 = v^2 = 1$. The ring has four maximal ideals with index of stability 1, which are given in the following:

$m_1 =< u+1, v+1 >$, $m_2 =< u+1, v-1 >$,

$m_3 =< u-1, v+1 >$, and $m_4 =< u-1, v-1 >$.

And the map $\psi$ defined by

$$\psi : R \longrightarrow \frac{R}{m_1} \times \frac{R}{m_2} \times \frac{R}{m_3} \times \frac{R}{m_4} \cong F_q^4 \qquad x \mapsto (x+m_1, x+m_2, x+m_3, x+m_4)$$

is a canonical isomorphism. According to A.Dertli et al. ([12]) we recall the following definitions

**Definition 2.1.** The Frobenius automorphism $\theta$ acting on $F_q$ defined by $\theta(y) = y^{p^t}$ for $y \in F_q$ induces on automorphism of the ring $R$ as: $\forall \alpha = r_1 + r_2 u + r_3 v + r_4 uv \in R$, $\theta(\alpha) = r_1^{p^t} + r_2^{p^t} u + r_3^{p^t} v + r_4^{p^t} uv$. then by the help of this $R$-authomorphism, we can define a ring as :
$R[y, \theta] = \{r_0 + r_1 y + r_2 y^2 + \cdots + y^n r_n^n : r_i \in R\}$, $1 \leq i \leq n$ with the multiplication denoted by $\bullet$ such that $(ry^i) \bullet (ly^j) = r\theta^i(l^{i+j})$, and by the usual addition of polynomials ring, $R[y, \theta]$ is called skew polynomial ring which is non-commutative unless $\theta$ is identity.

**Definition 2.2.** The linear code $C$ of length $n$ over $R$ is called skew $\alpha$-constacyclic with the automorphism $\theta$, where $\alpha$ is unit of $R$ fixed by $\theta$ if $\forall r = (r_1, r_2, \cdots, r_n) \in C$ $\Rightarrow \tau(r) = (\alpha\theta((r_n)), \theta(r_1), \cdots, \theta(r_{n-1})) \in C$. In particular, if $\alpha = 1$, then $C$ is called skew cyclic codes, and if $\alpha = -1$ then $C$ is said to be skew negacyclic code over $R$.

While on another side, if $\alpha = 1$, assuming $\tau(r) = (\theta((r_n)), \theta(r_1), \cdots, \theta(r_{n-1})) = \sigma(r)$.

**Definition 2.3.** Let $C$ be a linear code of length $ln$ over $F_q$. If $\pi_l(C) = C$, since $\pi_l$ defined by

$$\pi_l(r^1 \mid r^2 \mid \cdots \mid r^l) = (\sigma(r^1) \mid \sigma(r^2) \cdots \mid \sigma(r^l)), \text{ for } r^i \in F_q^n,\ 1 \leq i \leq l$$

, then $C$ is named by skew quasi-cyclic code.

**Definition 2.4.** Set $C^\perp = \{\vec{y} = (y_1, \cdots, y_n) \in R^n \mid \sum_{i=1}^n z_i y_i = 0, \forall \vec{z} = (z_1, \cdots, z_n) \in C\}$, then $C^\perp$ the dual code of $C$.

# 3 Gray map and Gray image of skew cyclic code over $R$

According to the fundamental result on the decomposition of modules, we have the following lemma:

**Lemma 3.1.** *([13]) Let $R$ be a finite ring and $I_1, I_2, \cdots, I_n$ be ideals of $R$. The following statements are equivalent:*

- $R = I_1 \bigoplus I_2 \bigoplus, \cdots, \bigoplus I_n$.
- *There exists a unique family of idempotents $(\beta_i)_{1 \leq i \leq n}$ such that $\beta_i^2 = \beta_i$, $\beta_i \beta_j = 0$ for $i \neq j$, $\sum_{i=1}^{n} \beta_i = 1$ and $I_i = \beta_i R$, $\forall i, j \in \{1, \cdots n\}$.*

By [12] we recall the following family idempotent is given by $\beta_1, \beta_2, \beta_3, \beta_4$ where,
$\beta_1 = 1 + u + v + uv$, $\beta_2 = 1 + u - v - uv$,
$\beta_3 = 1 - u + v - uv$, $\beta_4 = 1 - u - v + uv$.
Any element of $a$ of $R$ can be written as

$$a = a_1 + a_2 u + a_3 v + a_4 uv =$$
$$\beta_1(a_1+a_2+a_3+a_4)+\beta_2(a_1+a_2-a_3-a_4)+\beta_3(a_1-a_2+a_3-a_4)+\beta_4(a_1-a_2-a_3+a_4).$$

Let $C$ be a linear code of length $n$ over $R$, and :
$C_1 = \{a_1 + a_2 + a_3 + a_4 \in F_q^n || a_1 + a_2 u + a_3 v + a_4 uv \in C\}$,
$C_2 = \{a_1 + a_2 - a_3 - a_4 \in F_q^n | a_1 + a_2 u + a_3 v + a_4 uv \in C\}$,
$C_3 = \{a_1 - a_2 + a_3 - a_4 \in F_q^n | a_1 + a_2 u + a_3 v + a_4 uv \in C\}$,
$C_4 = \{a_1 - a_2 - a_3 + a_4 \in F_q^n | a_1 + a_2 u + a_3 v + a_4 uv \in C\}$.
Assume that $C = \beta_1 C_1 + \beta_2 C_2 + \beta_3 C_3 + \beta_4 C_4$.

**Lemma 3.2.** *if $G_1, G_2, G_3, and G_4$ are generator matrices of Linear codes $C_1, C_2, C_3, and C_4$ respectively, then the generator matrix of the linear code $C$ which is defined above has the following generator matrix*

$$\begin{pmatrix} \beta_1 G_1 \\ \beta_2 G_2 \\ \beta_3 G_3 \\ \beta_4 G_4 \end{pmatrix}$$

*Let $\eta$ be the map:*

$\eta : \frac{R[y,\theta]}{(y^n-1)} \rightarrow \frac{R[y,\theta]}{y^n+1)}$

$h(y) \mapsto h(-y)$

**Theorem 3.3.** *if $n$ is odd then $\eta$ is a left $R$-module isomorphism.*

*Proof.* Firstly, we have to prove that $\eta$ is well-defined.

If $h_1(y) \equiv h_2(y) \pmod{)}(\mathrm{y}^n - 1) \Longleftrightarrow \exists p(y) \in R[y, \theta]$

$\Longleftrightarrow h_1(y) = h_2(y) + p(y)(y^n - 1)$

$\Longleftrightarrow h_1(-y) = h_2(-y) + p(-y)((-y)^n - 1)$

$\Longleftrightarrow \eta(h_1(y)) \equiv \eta(h_2(y))$ (since $n$ is odd). Hence it can be concluded that $\eta$ is well-defined.

Secondly, $\eta$ is $R$-module monomorphism injective, and $\frac{R[y,\theta]}{(y^n-1)}, \frac{R[y,\theta]}{(y^n+1)}$ are finite rings which implies that $\eta$ is a left $R$-module isomorphism. □

By theorem 3.3 one can obtain the below corollary:

**Corollary 3.4.** *suppose that $C$ is a linear code of odd length $n$. Then $C$ is skew-cyclic code with respect to the automorphism $\theta$ over $R$ if and only if $\eta(C)$ is skew-negacyclic code over $R$.*

*Proof.* if $C$ is skew-cyclic code over $R \Leftrightarrow C$ is an ideal of $\frac{R[y,\theta]}{(y^n-1)}$ since $n$ is odd (by 3.3) $\eta(C)$ is also an ideal of $\frac{R[y,\theta]}{y^n+1)} \Leftrightarrow \eta(C)$ is skew-negacyclic code over $R$. □

**Lemma 3.5.** *Let $f_\pi$ the permutation version of Gray map from $R^n$ to $F_q^{4n}$ given by*

$f_\pi(r = (r_0, r_1, \cdots, r_{n-1})) = (\vec{a}, \vec{b}, \vec{c}, \vec{d})$ *with*

$\vec{a} = (r_0^0 + r_0^1 + r_0^2 + r_0^3, r_1^0 + r_1^1 + r_1^2 + r_1^3, \cdots, r_{n-1}^0 + r_{n-1}^1 + r_{n-1}^2 + r_{n-1}^3)$,

$\vec{b} = (r_0^0 + r_0^1 - r_0^2 - r_i^3, r_1^0 + r_1^1 - r_1^2 - r_1^3, \cdots, r_{n-1}^0 + r_{n-1}^1 - r_{n-1}^2 - r_{n-1}^3)$,

$\vec{c} = (r_0^0 - r_0^1 + r_0^2 - r_i^3, r_1^0 - r_1^1 + r_1^2 - r_1^3, \cdots, r_{n-1}^0 - r_{n-1}^1 + r_{n-1}^2 - r_{n-1}^3)$,

$\vec{d} = (r_0^0 - r_0^1 - r_0^2 + r_0^3, r_1^0 - r_1^1 - r_1^2 + r_1^3, \cdots, r_{n-1}^0 - r_{n-1}^1 - r_{n-1}^2 + r_{n-1}^3))$, *where* $r_i = r_i^0 + r_i^1 u + r_i^2 v + r_i^3 uv \in R$ *for* $0 \leq i \leq n-1$.

*Let* $\pi_4$ *be the operator shift of quasi-cyclic, and* $\sigma$ *is the skew-cyclic shift which are defined in preliminary. Then we have* $f_\pi \sigma = \pi_4 f_\pi$

*Proof.* Let $r = (r_0, r_1, \cdots, r_{n-1}) \in R$ such that

$r_i = r_i^0 + r_i^1 u + r_i^2 v + r_i^3 uv$, for $0 \leq i \leq n-1$

$f_\pi \sigma(r) = f_\pi(\theta(r_{n-1}), \theta(r_0), \cdots, \theta(r_{n-2})$, with $\theta(r_i) = r_i^{0p^t} + r_i^{1p^t} u + r_i^{2p^t} v + r_i^{3p^t} uv$ for $0 \leq i \leq n-1$.

On another side $\pi_4 f_\pi(r) = \pi_4(r_0^0 + r_0^1 + r_0^2 + r_0^3, r_1^0 + r_1^1 + r_1^2 + r_1^3, \cdots, r_{n-1}^0 + r_{n-1}^1 + r_{n-1}^2 + r_{n-1}^3,$

$r_0^0 + r_0^1 - r_0^2 - r_i^3, r_1^0 + r_1^1 - r_1^2 - r_1^3, \cdots, r_{n-1}^0 + r_{n-1}^1 - r_{n-1}^2 - r_{n-1}^3,$

$r_0^0 - r_0^1 + r_0^2 - r_i^3, r_1^0 - r_1^1 + r_1^2 - r_1^3, \cdots, r_{n-1}^0 - r_{n-1}^1 + r_{n-1}^2 - r_{n-1}^3,$

$r_0^0 - r_0^1 - r_0^2 + r_0^3, r_1^0 - r_1^1 - r_1^2 + r_1^3, \cdots, r_{n-1}^0 - r_{n-1}^1 - r_{n-1}^2 + r_{n-1}^3) = f_\pi \sigma(r)$ □

**Theorem 3.6.** *A linear code* $C$ *of length* $n$ *is skew cyclic over* $R$ *if and only if* $f_\pi(C)$ *is a skew quasi-cyclic code of length* $4n$ *over* $F_q$ *of index* 4.

*Proof.* Let $C$ be a linear code of length $n$. Suppose that $C$ is skew cyclic code i.e $\sigma(C) = C$ then $f_\pi(\sigma(C)) = f_\pi(C)$.By lemma 3.5 we have $\pi_4(f_\pi(C)) = f_\pi(\sigma(C))$, which implies that $\pi_4(f_\pi(C)) = f_\pi(C)$, i.e, $f_\pi(C)$ is skew quasi-cyclic code of length $4n$.

Conversely, let $f_\pi(C)$ be a skew quasi-cyclic code of index 4. By applying lemma 3.5 which gives us $\pi_4 f_\pi(C) = f_\pi(C)$, we can get $f_\pi \sigma(C) = f_\pi(C)$.

As $f_\pi$ is injective, then $\sigma(C) = C$, which implies $C$ is skew cyclic code. □

# 4 Skew constacyclic codes over $R$

In this part we have described the decomposition of skew $\alpha$-constacyclic codes over $R$, since $\alpha$ is unit of $R$ fixed by $\theta$.

S. Jitman et al.([14]) have proved that a linear code of length $n$ is skew $\lambda$-constacyclic code over finite ring $M$ if and only if the skew representation polynomial of $C$ is a left ideal of the ring $\frac{M[x,\Theta]}{(x^n-\lambda)}$.

Any element $\alpha_0 + \alpha_1 u + \alpha_2 v + \alpha_3 uv \in R$ can be uniquely expressed as

$$\alpha_0 + \alpha_1 u + \alpha_2 v + \alpha_3 uv = (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)\beta_1 + (\alpha_0 + \alpha_1 - \alpha_2 - \alpha_3)\beta_2 + (\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3)\beta_3 + (\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3)\beta_4.$$

Now the decomposition of skew $\alpha$-constacyclic codes over $R$ can be given as following

**Theorem 4.1.** *Let $\alpha = \alpha = \alpha_0 + \alpha_1 u + \alpha_2 v + \alpha_3 uv$ be a unit of $R$ fixed by $\theta$. suppose that $C$ is a linear code over $R$ can be written as a direct sum of linears codes as follow:*

*$C = \beta_1 C_1 \oplus \beta_2 C_2 \oplus \beta_3 C_3 \oplus \beta_4 C_4$, where $C_i$, for $1 \leq i \leq 4$ are linear codes over $F_q$. Then $C$ is skew $\alpha$-constacyclic code with automorphism $\theta$ over $R$ if and only if $C_1$ is skew $[\alpha_0+\alpha_1+\alpha_2+\alpha_3]$-constacyclic code over $F_q$, $C_2$ is skew $[\alpha_0+\alpha_1-\alpha_2-\alpha_3]$-constacyclic code over $F_q$, $C_3$ is skew $[\alpha_0-\alpha_1+\alpha_2-\alpha_3]$-constacyclic code over $F_q$, and $C_4$ is skew $[\alpha_0-\alpha_1-\alpha_2+\alpha_3]$-constacyclic code over $F_q$.*

*Proof.* Let $r = (r_0, r_1, \cdots, r_{n-1}) \in C$ such that $r_i = \beta_1 a_i + \beta_2 b_i + \beta_3 c_i + \beta_4 d_i$, for $0 \leq i \leq n-1$.

Let $a = (a_0, a_2, \cdots, a_{n-1})$, $b = (b_0, b_2, \cdots, b_{n-1})$, $c = (c_0, c_2, \cdots, c_{n-1})$, and $d = (d_0, d_2, \cdots, d_{n-1})$ then we have $a \in C_1$, $b \in C_2$, $c \in C_3$, and $d \in C_4$.

Suppose $C_1$ is skew $[\alpha_0+\alpha_1+\alpha_2+\alpha_3]$-constacyclic $C_2$ is skew $[\alpha_0+\alpha_1-\alpha_2-\alpha_3]$-constacyclic, $C_3$ is skew $[\alpha_0-\alpha_1+\alpha_2-\alpha_3]$-constacyclic, $C_4$ is skew $[\alpha_0-\alpha_1-\alpha_2+\alpha_3]$-constacyclic codes over $F_q$.

Then $\tau(r) = ([\alpha_0 + \alpha_1 u + \alpha_2 v + \alpha_3 uv]\theta(r_{n-1}), \theta(r_0), \cdots, \theta(r_{n-2}))$
$= ([\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3]\theta(a_{n-1}), \theta(a_0), \cdots, (a_{n-2}))\beta_1$
$+ [\alpha_0 + \alpha_1 - \alpha_2 - \alpha_3](\theta(b_{n-1}), \theta(b_0), \cdots, \theta(b_{n-2}))\beta_2$
$+ ([\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3]\theta(c_{n-1}), \theta(c_0), \cdots, \theta(c_{n-2}))\beta_3$
$+ ([\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3]\theta(d_{n-1}), \theta(d_0), \cdots, \theta(d_{n-2}))\beta_4)$.
So $\tau(r) \in C$ , for $r \in C$.Then $C$ is skew $\alpha$-constacyclic code over $R$. Conversely, let $a = (a_0, a_2, \cdots, a_{n-1}) \in C_1$, $b = (b_0, b_2, \cdots, b_{n-1}) \in C_2$, $c = (c_0, c_2, \cdots, c_{n-1}) \in C_3$, and $d = (d_0, d_2, \cdots, d_{n-1}) \in C_4$.
Suppose that $r_i = \beta_1 a_i + \beta_2 b_i + \beta_3 c_i + \beta_4 d_i$, for $0 \leq i \leq n-1$. Since $C$ is skew $\alpha$-constacyclic, then

$$\begin{gathered}\tau(r) \in C \Longrightarrow ([\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3]\theta(a_{n-1}), \theta(a_0), \cdots, \theta(a_{n-2})) \in C_1,\\ [\alpha_0 + \alpha_1 - \alpha_2 - \alpha_3](\theta(b_{n-1}), \theta(b_0), \cdots, \theta(b_{n-2})) \in C_2,\\ ([\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3]\theta(c_{n-1}), \theta(c_0), \cdots, \theta(c_{n-2})) \in C_3, \text{ and}\\ ([\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3]\theta(d_{n-1}), \theta(d_0), \cdots, \theta(d_{n-2}))\beta_4) \in C_4.\end{gathered}$$

We can conclude that $C_1$ is skew $([\alpha_0+\alpha_1+\alpha_2+\alpha_3]$-constacyclcic code, $C_2$ is skew $([\alpha_0+\alpha_1-\alpha_2-\alpha_3]$-constacyclcic code, $C_3$ is skew $([\alpha_0-\alpha_1+\alpha_2-\alpha_3]$-constacyclcic code, and $C_4$ is skew $([\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3]$-constacyclcic code over $F_q$. □

**Lemma 4.2.** *([15]) Let $R$ be a finite ring and $R^\times$ is its group of units. If $R$ can be written as a direct sum as following $R = R_1 \oplus R_2 \oplus \cdots \oplus R_m$ of rings $R_i$ for $\leq i \leq m$, then $R^\times$ will be written as $R^\times = R_1^\times \oplus R_2^\times \oplus \cdots \oplus R_k^\times$.*

Then by Lemma 4.2 we have concluded the following corollary:

**Corollary 4.3.** *Let $\alpha = \alpha_0+\alpha_1 u+\alpha_2 v+\alpha_3 uv$ be an element of $R$. Then $\alpha$ can be a unit in $R$ if and only if $(\alpha_0+\alpha_1+\alpha_2+\alpha_3)$, $(\alpha_0+\alpha_1-\alpha_2-\alpha_3)$, $(\alpha_0-\alpha_1+\alpha_2-\alpha_3)$, and $(\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3)$ are non zero elements of $F_q$.*

By Lemma [3.1] in [14].

**Corollary 4.4.** *Let $C = \beta_1 C_1 \oplus \beta_2 C_2 \oplus \beta_3 C_3 \oplus \beta_4 C_4$ where $C_i$, $1 \leq i \leq 4$ are linear codes over $F_q$. Then $C$ is skew $\alpha$-constacyclic code with automorphism $\theta$. Then its dual $C^{\perp} = \beta_1 C_1^{\perp} \oplus \beta_2 C_2^{\perp} \oplus \beta_3 C_3^{\perp} \oplus \beta_4 C_4^{\perp}$ is skew $\alpha^{-1}$-constacyclic codes over $R$, where $C_1^{\perp}$, $C_2^{\perp}$, $C_3^{\perp}$, and $C_4^{\perp}$ are skew $[\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3]^{-1}$-constacyclic, skew $[\alpha_0 + \alpha_1 - \alpha_2 - \alpha_3]^{-1}$-constacyclic, skew $[\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3]^{-1}$-constacyclic, and skew $[\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3]^{-1}$-constacyclic codes over $F_q$ of length $n$.*

*Proof.* we have $\alpha$ is an element of $R$ then $a = a_1 + a_2 u + a_3 v + a_4 uv = \beta_1(a_1 + a_2 + a_3 + a_4) + \beta_2(a_1 + a_2 - a_3 - a_4) + \beta_3(a_1 - a_2 + a_3 - a_4) + \beta_4(a_1 - a_2 - a_3 + a_4)$. Since $\alpha$ is an unit of $R$ wich equivalent to $(a_1 + a_2 + a_3 + a_4), (a_1 + a_2 - a_3 - a_4), (a_1 - a_2 + a_3 - a_4)$, and $(a_1 - a_2 - a_3 + a_4)$ are also units.

we have $C$ is skew $\alpha$-constacyclic code $\Longleftrightarrow$ (by theorem 4.1) $C_1$ is skew $([\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3]$-constacyclcic code, $C_2$ is skew $([\alpha_0 + \alpha_1 - \alpha_2 - \alpha_3]$-constacyclcic code, $C_3$ is skew $([\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3]$-constacyclcic code, and $C_4$ is skew $([\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3]$-constacyclcic code over $F_q$. And by Lemma [3.1] in [14] $\Longleftrightarrow$ $C_1^{\perp}$ is skew $[\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3]^{-1}$-constacyclic, and $C_2^{\perp}$ is skew $[\alpha_0 + \alpha_1 - \alpha_2 - \alpha_3]^{-1}$-constacyclic, $C_3^{\perp}$ skew $[\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3]^{-1}$-constacyclic, and $C_4^{\perp}$ is skew $[\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3]^{-1}$-constacyclic codes over $F_q$ of length $n$. By aplying again theorem 4.1 we have $C^{\perp}$ is skew $\alpha^{-1}$-constacyclic code over $R$. □

# 5 Conclution

In this paper, we have introduced the linear codes of length $n$ over $R = F_q + uF_q + vF_q + uvF_q$, where $u^2 = v^2 = 1$, and $q = 3^m$. It has also been proved that the Gray image of skew cyclic code of length $n$ over $R$ is skew quasi-cyclic code of length $4n$ over $F_q$.

It has also shown in detail that each linear code over $R$ can be defined by $\beta_1 C_1 \oplus \beta_2 C_2 \oplus \beta_3 C_3 \oplus \beta_4 C_4$(where $C_i$,$1 \leq i \leq 4$ are linear codes of length $n$ over $F_q$ ) is skew $\alpha$-constacyclic code if and only if $C_1$ is skew $[\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3]$-constacyclic, $C_2$ is skew $[\alpha_0 + \alpha_1 - \alpha_2 - \alpha_3]$-constacyclic, $C_3$ is skew $[\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3]$-constacyclic, $F_q$ and $C_4$ is skew $[\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3]$-constacyclic codes over $F_q$.

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