TEACHING THE DERIVATIVE THROUGH AN ALGEBRAIC–GEOMETRIC TANGENT MODEL AND DYNAMIC VISUALIZATION

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ABSTRACT

The derivative is often introduced through the limit of a difference quotient. This definition is mathematically rigorous; however, at the initial stage of learning it may be difficult for students, since it brings together several new ideas at once: increment, secant line, limit, tangent line, and instantaneous rate of change. As a result, students may learn differentiation rules mechanically without fully understanding the geometric and functional meaning of the derivative. This article presents an algebraic–geometric tangent model for the initial teaching of the derivative in the case of polynomial functions. Instead of beginning with the formal limit definition, the model first defines the tangent line by means of a double-root condition: the difference between a polynomial and its candidate tangent line has a double zero at the point of tangency. This condition uniquely determines the slope of the tangent line, and the resulting correspondence from each point to the corresponding slope is introduced as the derivative function. The model is based on concepts already familiar to students: function, graph, line, slope, divisibility of polynomials, and double root. The limit definition is not rejected in this approach; rather, it is obtained later as an analytic formalization of the already constructed tangent model. Dynamic visualization helps students connect the algebraic condition, the geometric tangent, and the transition to the classical limit formula. The article also discusses classroom and university teaching experience accumulated between 1986 and 1997, which indicates that this approach may strengthen students' conceptual understanding of the derivative as a means of describing the local behavior of a function.

KEYWORDS

derivative; tangent line; double root; polynomial function; limit-free approach; algebraic–geometric model; dynamic visualization; mathematics education; conceptual understanding.

1. INTRODUCTION

The derivative is one of the central concepts of calculus. It plays an important role both in the transition from school mathematics to higher mathematics and in the study of physics, engineering, and economics. Nevertheless, the teaching and understanding of the derivative remain among the serious conceptual difficulties in mathematics education.

In the standard course, the derivative is usually defined by means of the limit of a difference quotient:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

This formula is mathematically precise, but at the beginning of instruction it requires students to coordinate several new ideas at once: the increment of the independent variable, the increment of

the function, the difference quotient, the limiting process, the secant line, the tangent line, and the instantaneous rate of change. For many students this creates a considerable cognitive load. They may correctly apply a rule, for example
$(x^n)' = nx^{n-1}$,
but still be unable to explain why the derivative represents the slope of the tangent line or how it describes the local behavior of a function.

The aim of this article is to present an alternative initial route for teaching the derivative: an algebraic–geometric tangent model supported by dynamic visualization. The model is first developed for polynomial functions. Its main idea is that the tangent line can be defined not through a limiting process, but through an algebraic condition of contact.

Let $P(x)$ be a polynomial and let $x = a$ be a fixed point. Any line passing through the point $(a, P(a))$ can be written in the form

$$T_a(x) = P(a) + k(x - a).$$

Such a line is regarded as the tangent line to the graph of $P(x)$ at the point $x = a$ if the difference between the polynomial and this line is divisible by $(x - a)^2$, that is,

$$P(x) - T_a(x) = (x - a)^2 Q(x),$$

where $Q(x)$ is a certain polynomial. This condition algebraically expresses the fact that the line and the graph have double contact at the point $x = a$. At the same time, it uniquely determines the slope k. Therefore, to each point a there corresponds a unique slope $k(a)$ of the tangent line, and this correspondence
$a \mapsto k(a)$
is introduced as the derivative function; in other words,

$$P'(a) = k(a).$$

The approach presented here does not oppose the classical limit definition. Its purpose is for the student first to understand the algebraic and geometric meaning of the tangent line, then to see its slope as a function, and only afterwards to proceed to the limit formalization. Thus, the order of instruction is changed: first the tangent line is constructed, then its slope is interpreted as a functional correspondence, and finally the limit formula appears as an analytic expression of the same local linear behavior.

The pedagogical value of the model does not lie merely in the fact that it obtains familiar differentiation rules by another route. More importantly, students begin to explain the behavior of a function in terms of the slope of the tangent line. If $k(a) > 0$, the graph has a locally increasing direction; if $k(a) < 0$, it has a locally decreasing direction; and if $k(a) = 0$, the tangent line is horizontal, which naturally prepares the idea of a maximum or a minimum. Such reasoning may emerge before students formally study the theorems concerning increasing and decreasing functions and extrema.

Dynamic visualization provides significant support for this process. By changing the point a and the slope k, students can observe when the double-root condition is satisfied, how the tangent line is singled out, and how the correspondence $a \mapsto k(a)$ changes. In this way, the algebraic, geometric, and analytic representations of the derivative are integrated into a visible and accessible form.

In this article, the model is deliberately restricted to polynomial functions. It is precisely within this class that the double-root condition provides a clear and elementary algebraic criterion. The extension to other elementary functions, as well as the verification of the model through contemporary empirical methods, may be considered as a subjects for further research.

The article is organized as follows. Section 2 presents the cognitive–didactic foundation of the model and explains why the construction begins from students' prior knowledge. Section 3 develops the algebraic–geometric tangent model for polynomial functions, proves the uniqueness of the tangent slope, introduces the derivative as a slope function, and illustrates the approach through basic examples and differentiation rules. Section 4 explains the transition from the constructed tangent model to the classical limit formula. Section 5 discusses the pedagogical implementation of the model, the role of dynamic visualization, assessment of conceptual understanding, and classroom and university experience. Section 6 summarizes the main conclusions and indicates directions for further research.

2. COGNITIVE–DIDACTIC FOUNDATION

The difficulty of teaching the derivative cannot be explained only by the mathematical nature of the concept itself. It is also important to consider the sequence in which students encounter new ideas and the extent to which this process relies on their prior knowledge. When, at the very beginning of instruction, increment, difference quotient, limit, secant line, and tangent line are introduced simultaneously, students' attention often shifts toward formal operations, while the meaningful understanding of the concept becomes weaker.

According to cognitive load theory, working memory has a limited capacity. Therefore, at the initial stage of learning a complex concept, it is desirable to choose an approach that builds new knowledge on representations already familiar to students. In the case of the derivative, such prior knowledge includes the graph of a function, the equation of a line, the slope coefficient, the concept of a root, the double root, and divisibility of polynomials. These concepts provide a sufficient basis for constructing the idea of the tangent line first by an algebraic–geometric route, without the preliminary use of limits.

The initial question of this approach is the following: when is a line tangent to the graph of a polynomial at a given point? The fact that a line merely passes through a given point of the graph is not sufficient, since infinitely many such lines can be drawn. In order to single out the tangent line, a stronger condition is needed. Such a condition is the double-root condition. The line

$$T_a(x) = P(a) + k(x - a)$$

is regarded as the tangent to the graph of $P(x)$ at the point $x = a$, if the difference $P(x) - T_a(x)$ has a double root at this point, that is,

$$P(x) - T_a(x) = (x - a)^2 Q(x).$$

This criterion is both algebraic and geometric. Algebraically, it means divisibility by $(x - a)^2$; geometrically, it means not a simple intersection of the graph and the line, but contact. Thus, the idea of the tangent line arises naturally from algebraic concepts already known to students.

The logical path of instruction may be formulated as follows:

prior knowledge → double root → tangent line → slope k(a) → derivative function.

In this sequence, the derivative is not given to the student as a ready-made formula. It appears as a result: if, at each point, the slope of the tangent line is uniquely determined, then to each a there corresponds a certain k(a). It is precisely this correspondence that is defined as the derivative function.

In this way, students develop not only procedural knowledge, but also conceptual knowledge. They see that the derivative is not merely a rule for calculation; it is a function that describes the local direction of a given graph. This creates a natural foundation for subsequent topics: increasing and decreasing functions, extrema, and the analysis of graphs.

3. THE ALGEBRAIC–GEOMETRIC TANGENT MODEL FOR POLYNOMIAL FUNCTIONS

3.1. The Tangent Line as a Double-Root Condition

For polynomial functions, the tangent line can be defined without the preliminary use of limits. In this model, the algebraic basis of the tangent line is the double-root condition.

Let a polynomial P(x) and a point x = a be given. Any line passing through this point may be written in the form
$T_a(x) = P(a) + k(x - a)$,

where k is the slope coefficient of the line. For every value of k, this line passes through the point (a, P(a)); therefore, the difference $P(x) - T_a(x)$ vanishes at x = a. However, this only means that the line and the graph have a common point. In order to single out the tangent line, a stronger condition is required:
$P(x) - T_a(x) = (x - a)^2 Q(x)$,

where Q(x) is a certain polynomial.

This condition means that x = a is a double root of the difference $P(x) - T_a(x)$. Algebraically, this is expressed by divisibility by $(x - a)^2$, while geometrically it expresses contact between the line and the graph. Thus, the geometric idea of the tangent line is transformed into a precise algebraic criterion.

On the basis of this criterion, one can show that, for each fixed a, the corresponding k exists and is unique. Indeed, suppose that two different slopes k_1 and k_2 satisfy the double-root condition. Then we have two lines:
$T_1(x) = P(a) + k_1(x - a)$,
$T_2(x) = P(a) + k_2(x - a)$.

In both cases, the difference $P(x) - T_i(x)$ is divisible by $(x - a)^2$. Therefore, their difference must also be divisible by $(x - a)^2$. But
$T_2(x) - T_1(x) = (k_2 - k_1)(x - a)$.

This expression is divisible only by (x − a), and not by $(x - a)^2$, if $k_2 - k_1 \neq 0$. Consequently, it is necessary that $k_2 - k_1 = 0$, that is, $k_1 = k_2$. Hence, the slope of the tangent line is uniquely determined.

This reasoning is important for students because it shows that the idea of the derivative does not begin with a ready-made formula. First, the unique line that has double contact with the graph at the given point is singled out; then the slope of this line becomes the basis of a new concept.

3.2. The Derivative as a Slope Function

Since for each point x = a the slope k of the tangent line is uniquely determined, we obtain the correspondence
$a \mapsto k(a)$.

This correspondence is called the derivative function of P(x), and we write
$P'(a) = k(a)$.

From the instructional point of view, this sequence is fundamental:
$x = a \rightarrow$ tangent line $T_a(x) \rightarrow$ slope $k(a) \rightarrow$ derivative $P'(a)$.

In this approach, the derivative is not introduced initially as a limit. It appears as a functional correspondence: to each point there corresponds the unique slope of the tangent line. This is the initial meaning of the derivative.

Thus, students see that the derivative is not merely a rule for calculation. It is a function that describes how the local direction of a given function changes from point to point. This is especially important for subsequent topics — increasing and decreasing functions, extrema, and graph construction — because their understanding is directly connected with the sign and change of the tangent slope.

3.3. Example: $P(x) = x^2$

Let us find the slope of the tangent line to the function $P(x) = x^2$ at the point x = a. The line passing through this point is
$T_a(x) = a^2 + k(x - a)$.

Consider the difference
$x^2 - [a^2 + k(x - a)]$.
Let us simplify it:
$x^2 - [a^2 + k(x - a)] = x^2 - a^2 - k(x - a) =$
$= (x - a)(x + a) - k(x - a) = (x - a)(x + a - k)$.
In order for this expression to be divisible by $(x - a)^2$, the second factor $x + a - k$ must vanish at x = a. Therefore,$a + a - k = 0$,from which we obtain$k = 2a$.Hence,
$P'(a) = 2a$,that is,$(x^2)' = 2x$.

This result has been obtained without using limits — only by applying the double-root condition. In this example, students see not the application of a ready-made rule, but how the tangent line is singled out and how its slope becomes the derivative.

3.4. The General Case: $P(x) = x^n$

Now consider the general power function
$P(x) = x^n$.
The line passing through the point x = a is written in the form
$T_a(x) = a^n + k(x - a)$.
We require that the difference $x^n - T_a(x)$ be divisible by $(x - a)^2$. We have

$x^n - T_a(x) = x^n - a^n - k(x - a)$.
Using the well-known decomposition,
$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + xa^{n-2} + a^{n-1})$,
we obtain
$x^n - a^n - k(x - a) =$
$= (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + xa^{n-2} + a^{n-1} - k)$.
In order for this expression to be divisible by $(x - a)^2$, the expression in parentheses must vanish at $x = a$. Substituting $x = a$, we get
$a^{n-1} + a^{n-1} + ... + a^{n-1} - k = 0$.
Since there are n such terms, we obtain$na^{n-1} - k = 0$.Therefore,$k = na^{n-1}$.
Hence,
$P'(a) = na^{n-1}$,that is,$(x^n)' = nx^{n-1}$.

Thus, the rule for differentiating a power function is obtained not as a formula to be memorized, but as a result of the algebraic construction of the tangent line. Students see that the formula follows from the double-root criterion and therefore has a geometric meaning.

3.5. The Sum Rule as a Model of Algebraic–Geometric Proof

The algebraic–geometric tangent model also makes it possible to obtain the basic rules of differentiation from the same double-root criterion. To illustrate this, let us consider the sum rule as an example, in a form similar to which other rules for polynomials may subsequently be proved.

Remark. If P(x) and R(x) are polynomials, then their sum S(x) = P(x) + R(x) is again a polynomial, and its derivative may also be defined directly by the double-root criterion. However, here we consider P(x) and R(x) separately in order to see how the slope of the tangent line to the sum is connected with the slopes of the tangent lines to the addends. This is precisely what creates a model of algebraic–geometric proof.

Let P(x) and R(x) be polynomial functions, and let
$S(x) = P(x) + R(x)$.
If, for P(x) and R(x) at the point $x = a$, we have
$P(x) - [P(a) + P'(a)(x - a)] = (x - a)^2Q_1(x)$,
$R(x) - [R(a) + R'(a)(x - a)] = (x - a)^2Q_2(x)$,
then by adding these equalities we obtain
$S(x) - [S(a) + (P'(a) + R'(a))(x - a)] = (x - a)^2[Q_1(x) + Q_2(x)]$.
Therefore, the slope of the tangent line to S(x) is $P'(a) + R'(a)$, that is,
$(P + R)' = P' + R'$.

Thus, the sum rule follows from the same double-root criterion by which the derivative itself was defined. This proof may be used as a model: by means of analogous algebraic–geometric reasoning, students or university students can independently obtain the product rule, the quotient rule, and the chain rule for polynomials. Such a task checks not only computational ability, but also the extent to which students can transfer the knowledge they have acquired to a new situation and construct a new argument.

3.6. The Non-Existence of the Derivative for f(x) = |x|

In order to strengthen the concept of the derivative, it is important to consider an example where the derivative does not exist. Let us consider the function
$f(x) = |x|$

at the point x = 0.
If x > 0, then |x| = x, and the local slope of the graph is +1.
If x < 0, then |x| = −x, and the local slope of the graph is −1.

Thus, at the point x = 0 the graph has two different local directions. From the left, the slope is −1, while from the right, the slope is +1. Therefore, a unique tangent line does not exist at this point, and consequently the derivative does not exist at this point either.
This example shows that the derivative is not merely the application of a formula. It exists only when the graph has a uniquely determined local direction at the given point.

4. TRANSITION TO THE LIMIT: ANALYTIC FORMALIZATION OF THE TANGENT MODEL

4.1. The Basic Geometric Picture

Consider the graph of the function y = f(x) shown in Figure 1 and two points on it:
A(x_0, f(x_0))and B(x_0 + Δx, f(x_0 + Δx)).

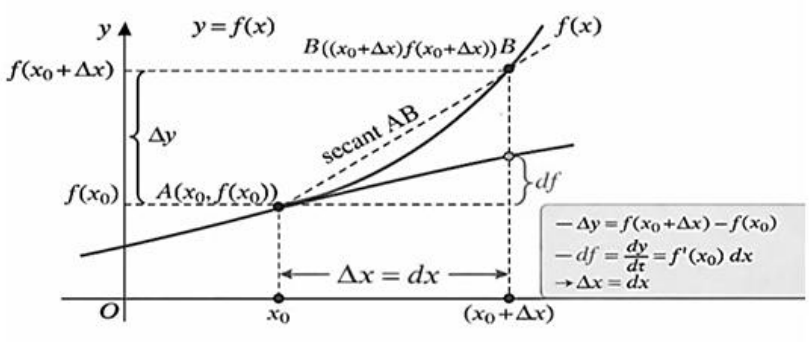

Figure 1

The line passing through these two points is a secant line. Its slope is calculated by the ratio

$$\frac{[f(x_0 + \Delta x) - f(x_0)]}{\Delta x}$$

When Δx decreases and the point B approaches the point A, the secant line gradually approaches the line which we call the tangent to the graph at the point A. In this picture, the slope of the tangent line may be regarded as the limiting position of the slope of the secant line.

However, in the present model, the tangent line is not initially defined by means of a limit. It has already been constructed algebraically — by the double-root condition. Therefore, the limit formula is introduced not as the initial definition, but as an analytic formalization of the previously constructed tangent model.

4.2. Linear Decomposition and Derivation of the Limit Formula

In Section 3, the derivative was defined by the algebraic criterion of the double root. We now show how the classical limit formula for the derivative is obtained from this model.
For a polynomial function, the tangent condition means that at the point x = x_0 we have the decomposition
$f(x) = f(x_0) + f'(x_0)(x - x_0) + (x - x_0)^2 Q(x)$,
where Q(x) is a certain polynomial.

If $x = x_0 + \Delta x$, then we obtain
$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + (\Delta x)^2 Q(x_0 + \Delta x)$.
Here the first two terms represent the linear approximation determined by the tangent line, while the last term is the remainder, which depends on the square order of Δx.
Move $f(x_0)$ to the left-hand side and divide by Δx:

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0) + R_1(\Delta x)$$

Where $R_1(\Delta x) = \Delta x \cdot Q(x_0 + \Delta x)$.As $\Delta x \to 0$, the expression $\Delta x \cdot Q(x_0 + \Delta x)$ tends to zero. Therefore, we obtain

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Thus, in this approach, the limit is not used as the initial definition. It is obtained as an analytic consequence of the already constructed algebraic–geometric tangent model.

4.3. The Unity of Three Levels

In this approach, three levels are naturally integrated.

At the geometric level, as the point B approaches the point A, the secant line approaches the tangent line, and the slope of the secant line approaches the slope of the tangent line.
At the algebraic level, the double-root condition, that is, divisibility by $(x - a)^2$, uniquely determines the tangent line.

At the analytic level, the decomposition of the increment and the behavior of the remainder term lead to the classical limit formula for the derivative.

These three levels create a conceptual bridge between the student's geometric intuition and the formalism of classical analysis. The instructional path moves from the simpler to the more general: first the algebraic criterion, then the geometric interpretation, and finally the analytic formula.

5. PEDAGOGICAL IMPLEMENTATION AND EMPIRICAL EXPERIENCE

5.1. The Instructional Purpose of the Model

The algebraic–geometric tangent model is intended for the initial teaching of the concept of the derivative. Its purpose is not to reject or replace the classical limit-based approach. On the contrary, the model creates a preparatory bridge through which students can move more naturally toward the formal concept of limit.

In this approach, the derivative is presented from the beginning not merely as a rule for calculation, but as a functional correspondence of tangent slopes. First, students understand what it means for a tangent line to exist at a given point and how its slope is determined; only afterwards do they learn the analytic formalization of this idea by means of a limit.

The model may be used in upper secondary school mathematics, in a first-year calculus course, and in teacher education courses. In all three cases, its main purpose is to construct a complex concept on the basis of students' prior knowledge.

5.2. Recommended Instructional Sequence

The implementation of the model in the teaching process may be organized in five consecutive stages.

At the first stage, prior knowledge is activated. Students recall the graph of a function, the equation of a line, the slope coefficient, the concept of a root, the double root, and divisibility of polynomials.

At the second stage, lines passing through a given point are considered. At the point x = a, all such lines may be written in the form
$T_a(x) = P(a) + k(x - a)$.
Students see that all these lines pass through the same point, but only one of them should be the tangent line. At this point the central question arises: what condition singles out this one line?
At the third stage, the double-root condition is introduced:
$P(x) - T_a(x) = (x - a)^2 Q(x)$.
Students test this condition on concrete examples and see that, in this way, k is uniquely determined.

At the fourth stage, the derivative is defined as a functional correspondence. For different values of a, students find the corresponding k(a) and notice that to each point there corresponds a unique slope of the tangent line. It is precisely this correspondence
$a \mapsto k(a)$
that is defined as the derivative function.

At the fifth stage, the transition to the limit formula is made. After the tangent model has been understood, the classical limit formula no longer appears as an initially obscure abstraction; rather, it appears as an analytic continuation of the already constructed algebraic–geometric model.

5.3. The Role of Dynamic Visualization

Dynamic visualization is an important component of the model, because it allows students to see the algebraic condition, the geometric picture, and the transition to the limit formula simultaneously.

First, students change k and observe when the condition
$P(x) - T_a(x) = (x - a)^2 Q(x)$
is satisfied. In this way, the algebraic criterion becomes visible: the tangent line is no longer merely a “touching” line drawn on the graph, but an object determined by a specific condition.

Second, students observe the transition from the secant line to the tangent line. When the second point of the graph approaches the first point, the slope of the secant line approaches the slope of the tangent line. This visually prepares the idea of the limit, but does not begin instruction with the formal definition of the limit.

Third, students change a and see how k(a) changes. At this moment, the derivative appears as a function — not as an isolated number, but as a correspondence that assigns to each point the slope of its tangent line.

On the interactive platform, both the tangent model constructed for polynomial functions and additional visual modules for exponential, logarithmic, and trigonometric functions are presented.

These modules are used not as part of the main proof of this article, but as a visual illustration of further extension and of the transition to the limit formula.

The interactive platform is available at:
https://kapanadze-theory.streamlit.app

5.4. Assessment of Conceptual Understanding

When this approach is used, assessment should not be limited only to computational tasks. It is necessary to include questions that check whether students see the connection between the algebraic condition, the tangent line, the slope, and the derivative function.

The following types of questions may be used:

1. Why is it not sufficient for a line simply to pass through a given point of the graph? What additional condition singles out the tangent line?
2. What does the condition$P(x) - [P(a) + k(x - a)] = (x - a)^2Q(x)$mean algebraically and geometrically?
3. Why is the derivative defined as the correspondence $a \mapsto k(a)$?
4. Why does the derivative of $f(x) = |x|$ not exist at the point $x = 0$?
5. How is the classical limit formula obtained from the decomposition of the increment?

Such questions assess not only computational ability, but also whether students are able to apply newly acquired knowledge in a new situation.

5.5. Empirical Basis: Research Stages and Classroom Experience

The model presented in this article was developed on the basis of long-term pedagogical experience and classroom observations. The research developed in three main stages.

Stage I: 1986–1989 — traditional instruction and identification of the problem.
During this period, the derivative was taught by means of the traditional limit-based approach. Observations involving approximately 100 students and first-year university students showed that they were able to apply standard formulas, but had difficulty understanding the derivative as a function and as a correspondence of tangent slopes. Their knowledge often remained at a formal level and was less effectively transferred to an understanding of the behavior of functions.

Stage II: 1990–1992 — development of the algebraic–geometric approach and comparative instruction.
At this stage, an approach based on the double-root criterion for the tangent line was developed, and pilot instruction was conducted in parallel groups. Each group included approximately 20 participants, and the instructional cycle consisted of approximately ten teaching hours.
In Group A, the traditional limit-based approach was used. Students mostly acquired formulas, but their conceptual understanding of the derivative remained weak.
In Group B, the algebraic–geometric approach was used. In this group, the derivative was more often perceived not merely as a rule for calculation, but as a functional correspondence of tangent slopes.

The assessment included not only computational tasks, but also conceptual questions: what the derivative means, why it is connected with the slope of the tangent line, and whether the derivative exists at every point.

Stage III: 1993–1997 and further development — university-level implementation.
In subsequent years, the approach was used in optional school courses, with students of pedagogical institutes, and with future teachers. In the university setting, three types of groups were used for comparison.

In Group A, instruction proceeded through the traditional limit-based approach. In this case, the limit formula was often used automatically, without meaningful explanation.

In Group B, the algebraic–geometric approach was used. In this group, the connection between the tangent line and the derivative function was more clearly visible.

In Group C, a mixed approach was used. In this case, the limit formula was more naturally connected with the previously constructed tangent model.

After 1997, controlled experiments did not continue in the same form; however, elements of the model were used in pedagogical practice, in work with teachers, and, in recent years, in tools for dynamic visualization.

5.6. Differences in Students' Reasoning Across Groups

In comparative instruction, what proved important was not only the number of correct answers, but also the type of reasoning used by students. Without having previously studied the full investigation of functions, all three groups were asked the following question: how can the behavior of a function be described in terms of the slope of the tangent line?

The majority of students in the traditional group could not formulate a clear answer to this question. For them, the derivative was mainly associated with a formula, calculation, or the formal notation of a limit; therefore, they had difficulty using it to explain the behavior of a function in a situation that had not yet been formally studied.

Students in the algebraic–geometric group, however, used geometric reasoning more naturally: if $k(a) > 0$, the tangent line has an increasing direction and the function is increasing; if $k(a) < 0$, the tangent line is decreasing and the function is decreasing; and if $k(a) = 0$, the tangent line is horizontal, which may indicate the existence of a maximum or a minimum.

The significance of these answers lies in the fact that students drew conclusions about the behavior of a function not by applying previously memorized theorems, but by relying on the geometric meaning of the slope of the tangent line. This demonstrates the main pedagogical value of the model: the derivative moves from being a rule for calculation to becoming a means of describing the local behavior of a function, which is an important indicator of conceptual understanding.

5.7. Limitations of the Research and Further Tasks

The empirical material presented here is based on classroom observations, comparative groups, and long-term pedagogical experience. The early stages of the research were conducted during a period when modern standardized instruments of educational research were not yet widely established. Therefore, the results should be regarded as classroom-pedagogical and historical empirical evidence, rather than as a contemporary large-scale statistical study.

Despite this limitation, the observations indicate that the algebraic–geometric approach supports the understanding of the derivative as a means of describing the local behavior of a function. A

further research task is to re-examine the model using a contemporary design: pre-tests and post-tests, interviews, control and experimental groups, as well as quantitative and qualitative analysis.

6. CONCLUSION

The algebraic–geometric tangent model presented in this article shows that the teaching of the derivative can begin in a way that is more accessible to students — through the double root, the tangent line, and functional correspondence. In this approach, the new concept is not given to the student as a ready-made formula; rather, it is constructed from familiar algebraic and geometric ideas.

The main mathematical idea of the model is the following: a line passing through the point $x = a$ on the graph of a polynomial is regarded as a tangent line when the difference $P(x) - T_a(x)$ is divisible by $(x - a)^2$. This condition uniquely determines the slope of the tangent line, $k(a)$, while the correspondence

$a \mapsto k(a)$

is defined as the derivative function.

From the instructional point of view, it is important that the limit is not used in this model as the initial definition. Students first understand the algebraic–geometric nature of the tangent line, then see the derivative as the function of tangent slopes, and only afterwards move to the classical limit formula. Dynamic visualization strengthens this process, because it simultaneously shows students the algebraic criterion, the geometric tangent line, and the transition to the limit.

Classroom and university experience indicates that students taught through the algebraic–geometric approach can more easily connect the derivative with the behavior of a function. By using the sign of the tangent slope, they explain increasing behavior, decreasing behavior, and the possible connection of a horizontal tangent line with a maximum or a minimum — even when these topics have not yet been formally studied. In this article, the model is restricted to polynomial functions; its extension to other elementary functions and its verification through contemporary empirical methods remain tasks for further research.

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