SPECTRAL ESTIMATE FOR STABLE SIGNALS WITH P-ADIC TIME AND OPTIMAL SELECTION OF SMOOTHING PARAMETER

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ABSTRACT

The spectral density of stable signals with p-adic times is already estimated under various conditions. The estimate is made by constructing a periodogram that is subsequently smoothed by a spectral window. It is clear that the convergence rate of this estimator depends on the bandwidth of the spectral window (called the smoothing parameter). This work gives a method to select the smoothing parameter in an optimal way, i.e. the estimator converges to the spectral density with the bestrate.

The method is inspired by the cross-validation method, which consists in minimizing the estimate of the integrated square error.

KEYWORDS

Spectral density, p-adic processes, alpha stable, random field.

1. INTRODUCTION

This work concerns the family of alpha stable random fields, which are known by infinite variance (infinite energy). These processes are widely used models for many phenomena in several fields such as physics, biology, electronics and electricity, hydrology, economics, communications and radar applications... see [1]-[12]. The paper [13] studied the estimation of the spectral density of these processes when they are continuous time, and in [14]-[15] when the process is discrete time. The article [16] extended this work to stable random fields with p-adic time.More specifically, the process has the following spectral representation:

$$X(t_1, t_2) = \int_{\mathrm{Q}_\mathrm{p}^2} \mathrm{e}^{\mathrm{i}<t_1\lambda_1 + t_2\lambda_2>} \mathrm{d}M(\lambda_1, \lambda_2);$$

$(t_1, t_2) \in \mathrm{Q}_\mathrm{p}^2$ where Q_p^2 is the field of p-adic numbers and M is a alpha stable random measure with a control measure m. The paper [16] studied the case where the measure m is continuous with respect to the Haar measure:

$\mathrm{d}m(x_1, x_2) = \Phi(x_1, x_2) d\mathcal{H}(x_1, x_2), \quad (x_1, x_2) \in \mathrm{Q}_\mathrm{p}^2$. The density function Φ is called the spectral density of the process X. The paper [16] constructed a modified periodogram by observing the process on the ball U_n. This periodogram was then smoothed by a spectral window in order to have an asymptotically unbiased consistent estimator of the spectral density. It is logical that the convergence rate of this estimator depends on the bandwidth of the spectral window (smoothing parameter).

The goal of this work is to propose a method for selecting the optimal smoothing parameter, allowing the best rate of convergence of the estimator. This method uses the technique of cross-validation, which has proven itself in various fields of statistics.

The choice of p-adic numbers is motivated by a great use in spatial discretization. P-adic numbers have provided solutions to many questions in physics, including questions related to string theory (related to the p-adic quantum domain) and those related to hierarchically structured fractal behaviors. They were also used in: turbulence theory, dynamical systems, statistical physics, biology, see [17]-[21]. The 2-adic number which is a special case used for computer design see [22]. Modeling in quantum mechanics used p-adic statistics [23]-[25] based on probabilistic calculations when the number of trials is infinitely large. Bernoulli's classical probabilities have been studied by [26]. The paper [27] focused on the p-adic probability theory of stochastic processes. The article [28] develops the theory of stochastic integrals with respect to p-adic Brownian motion. Using the p-adic differentiation operator, papers [29]-[32] developed the properties of the trajectories of a p-adic Wiener process.

The paper [34] set up the spectral theory and Fourier transforms of stationary p-adic processes:$\mathrm{X}(t)t \in \mathrm{Q_p}$ where $\mathrm{Q_p}$ is the field of p-adic numbers. He also proposed a spectral density estimator by constructing a periodogram in the same way as for real-time stationary processes. The paper [35] gives an estimatorfrom discrete-time observation : $\mathrm{X}(\tau_{\mathrm{k}})_{\mathrm{k}\in\mathrm{Z}}$ where $(\tau_{\mathrm{k}})_{\mathrm{k}\in\mathrm{Z}}$instantsof observation are taken from $\mathrm{Q_p}$, associated with a Poisson process.

This paper is structured as follows: section 2 gives the asymptotically unbiased and consistent estimate (smoothing of the periodogram) and the results (propositions 2.1-2.5) presented in [34]. Section 3 shows that the estimator converges in probability towards the spectral density (Proposition 2.6), gives the cross-validation criterion and Theorems 3.1-3.3 show that this criterion makes it possible to find the optimal smoothing parameter. Section 4 is reserved to the conclusion, potential applications and open research issues.

2. PERIODOGRAM AND SPECTRAL DENSITY ESTIMATION

Consider a process $\mathrm{X} = \left\{\mathrm{X}_{\mathrm{t}_1,t_2}/(t_1,t_2) \in \mathrm{Q}_\mathrm{p}^2\right\}$ where Q_p^2 is the field of p-adic numbers having the following integral representation

$$\mathrm{X}_{\mathrm{t}_1,t_2} \int_{\mathrm{Q}_\mathrm{p}^2} \mathrm{e}^{\mathrm{i}<t_1\lambda_1+t_2\lambda_2>} \mathrm{d}M(\lambda_1,\lambda_2) \quad (1)$$

$\forall (t_1,t_2) \in \mathrm{Q}_\mathrm{p}^2$ where M is a symmetric αstable $S\alpha\mathrm{S}$ random measure with independent and isotropic increments. There exists a control measure m that is defined by: $\mathrm{m}(\mathrm{A}\times B) = [\mathrm{M}(\mathrm{A}\times B), M(\mathrm{A}\times B)]_\alpha^{1/\alpha}$.

Assume that the measure m is absolutely continuous with respect to Haar measure: $\mathrm{d}m = \Phi(\mathrm{x}_1,x_2)\mathrm{d}\mathbb{H}(\mathrm{x}_1,x_2)$ where $\mathbb{H}$ is Haar measure.

The paper [13] gave an estimator of the density Φ, called the spectral density of the process when the process is continuous real time. In [14]-[15] the estimator is studied when the process and the random field are discrete time. The article [16] gives the estimator of the spectral density when the process is in p-adic time:$X=\{X_{t_1,t_2}/(t_1,t_2)\in Q_p^2\}$. For that, he takes the ball $U_n=\{(x,y)\in Q_{p;;}^2 | (x,y)|_p\le p^n\}$ as the observation of the process. He considers the following periodogram: For $(\lambda_1,\lambda_2)\in Q_p^2$

$$d_n(\lambda_1,\lambda_2)=A_n Re\int_{U_n} e^{-i<t_1\lambda_1+t_2\lambda_2>}p^{-n}h(t_1p^n,t_2p^n)X(t_1,t_2)d\mathcal{H}((t_1,t_2)) \quad (2)$$

$$H(\lambda_1,\lambda_2)=\int_{Z_p^2} h(t_1,t_2)e^{-i<t_1\lambda_1+t_2\lambda_2>}d\mathcal{H}(t_1,t_2)$$

$$B_\alpha=\int_{Q_p^2}|\mathcal{H}(\lambda_1,\lambda_2)|^\alpha d\mathcal{H}(\lambda_1,\lambda_2)<+\infty$$

$$H_n(\lambda_1,\lambda_2)=\left(\frac{p^{2n}}{B_\alpha}\right)^{\frac{1}{\alpha}}H(p^{-n}\lambda_1,p^{-n}\lambda_2) \ =A_nH(p^{-n}\lambda_1,p^{-n}\lambda_2)$$

Therefore, $A_n=\left(\frac{p^{2n}}{B_\alpha}\right)^{\frac{1}{\alpha}}$.

$$\int_{Q_p^2}|H_n(\lambda_1,\lambda_2)|^\alpha d\mathcal{H}(\lambda_1,\lambda_2)=\int_{Q_p^2}\frac{p^{2n}}{B_\alpha}|H(p^{-n}\lambda_1p^{-n}\lambda_2)|^\alpha d\mathcal{H}(\lambda_1,\lambda_2)$$

$$=\frac{p^{2n}}{B_\alpha}p^{-2n}\int_{Q_p^2}|H(v_1,v_2)|^\alpha d\mathcal{H}(v_1,v_2)$$

Since $|p^{-n}|_p=p^{+n}$, they obtain

$\int_{Q_p^2}|H_n(\lambda_1,\lambda_2)|^\alpha d\mathcal{H}(\lambda_1,\lambda_2)$=1

The following propositions 2.1-2.5 are proved in [16]

Proposition 2.1 Let

$\Psi_n(\lambda_1,\lambda_2)=^\Delta\int_{Q_p^2}|H_n(\lambda_1-u_1,\lambda_2-u_2)|^\alpha\Phi(u_1,u_2)d\mathcal{H}(u_1,u_2)$. *If* Φ*is a continuous and boundedfunction, then*$B_\alpha(\Psi_n(\lambda_1,\lambda_2)-\Phi(\lambda_1,\lambda_2))$ *converges to zero as n tends to infinity.*

Proposition 2.2 *Let* $(\lambda_1,\lambda_2)\in Q_p^2$ *the characteristic function of* $d_n(\lambda_1,\lambda_2), Eexp\{ird_n(\lambda_1,\lambda_2)\}$, *converges to* $exp\{-C_\alpha|r|^\alpha\Phi(\lambda_1,\lambda_2)\}$.

The periodogram is modified as follows:

$$I_n(\lambda_1,\lambda_2)C_{q,\alpha}|d_n(\lambda_1,\lambda_2)|^q, \quad (3)$$

$where$0<q<2 and the normalization constant is given by $C_{(q,\alpha)}=\frac{D_q}{F_{q,\alpha}C_\alpha^{q/\alpha}}$ where $D_q=\int\frac{1-cos(u)}{|u|^{1+q}}du$ and $F_{q,\alpha}=\int\frac{1-\exp(-|u|^\alpha)}{|u|^{1+q}}du$

$C_\alpha=(\alpha\pi)^{-1}\int_0^\pi|\cos(\theta)|^\alpha d\theta,$

Propositions 2.3 *Let* $(\lambda_1,\lambda_2)\in Q_p^2$, *then* $EI_n(\lambda_1,\lambda_2)=(\Psi_n(\lambda_1,\lambda_2))^{\frac{q}{\alpha}}$ *and* $I_n(\lambda_1,\lambda_2)$ *is an asymptotically unbiased estimator of the spectral density but not consistent*

$EI_n(\lambda_1,\lambda_2) - (\Phi(\lambda_1,\lambda_2))^{q/\alpha}$=*o(1) and* $VarI_n(\lambda_1,\lambda_2) - V_{\alpha,q}(\Phi(\lambda_1,\lambda_2))^2$, *with* $V_{\alpha,q} = \frac{C^2_{q,\alpha}}{C_{2q,\alpha}} - 1$.
In order to have an asymptotically and consistent estimate, we smooth the periodogram that was modified using a spectral window.

$$f_n(\lambda_1,\lambda_2) = \int_{Q_p^2} W_n(\lambda_1 - u_1, \lambda_2 - u_2) I_n(u_1,u_2) d\mathcal{H}(u_1,u_2)$$

where $W_n(x_1,x_2) = |M_n|_p W(x_1 M_n, x_2 M_n)$ such that

$$M_n \to \infty \ ; \frac{M_n}{n} \to 0; |M_n|_p \to 0 and \frac{|M_n|_p}{|p^n|_p} \to \infty. \qquad (4)$$

The function W is an even nonnegative function vanishing outside $[-1,1]^2$and $\int_{Q_p^2} W(v_1,v_2)d\mathcal{H}(v_1;v_2) = 1$.

Proposition 2.4 *Let* $(\lambda_1,\lambda_2) \in Q_p^2$ *and* $Bias(f_n(\lambda_1,\lambda_2)) = E[f_n(\lambda_1,\lambda_2)] - (\Phi(\lambda_1,\lambda_2))^{p/\alpha}$, *then* $Bias(f_n(\lambda_1,\lambda_2)) = o(1)$. *Moreover, if* Φ *verifies*$|\Phi(x_1,x_2) - \Phi(y_1,y_2)| \le cste|(x_1 - y_1, x_2 - y_2)|_p^{-k}$, *then,* $Bias(f_n(\lambda_1,\lambda_2)) = O\left(\frac{1}{|M_n|_p^{-\frac{kp}{\alpha}}}\right)$.

Proposition 2.5 *Let* (λ_1,λ_2) *be in* Q_p^2. *Assume that* $\Phi \in L^1_{Q_p^2}$. *Then* $Var(f_n(\lambda_1,\lambda_2)) = O(p^{-n}M_n^{-3n})$.

From propositions 2.4 and 2.5, we show in the following proposition that $(f_n(\lambda_1,\lambda_2))^{\frac{\alpha}{q}}$ converges to $\phi(\lambda_1,\lambda_2)$ in probability.

Proposition 2.6 Let λ_1,λ_2 *p-adic numbers such that* $\phi(\lambda_1,\lambda_2) > 0$. *Then,*$(f_n(\lambda_1,\lambda_2))^{\frac{\alpha}{q}}$*converges in probability to* $\phi(\lambda_1,\lambda_2)$.

Proof We show that $f_n(\lambda_1,\lambda_2)$ converges in mean quadratic to $\phi(\lambda_1,\lambda_2)^{\frac{q}{\alpha}}$. $E\left|f_n(\lambda_1,\lambda_2) - \phi(\lambda_1,\lambda_2)^{\frac{q}{\alpha}}\right|^2 = Ef_n(\lambda_1,\lambda_2) - \phi(\lambda_1,\lambda_2)^{\frac{q}{\alpha}})^2 + Var(f_n(\lambda_1,\lambda_2)$. Then, from proposition 2.3, $E\left|f_n(\lambda_1,\lambda_2) - \phi(\lambda_1,\lambda_2)^{\frac{q}{\alpha}}\right|^2$ converges to zero. Thus,$(f_n(\lambda_1,\lambda_2))^{\frac{\alpha}{q}}$ converges to $\phi(\lambda_1,\lambda_2)$ in probability.

It is obvious that the choice of $M_n$plays an important role, since the convergence rates depend on this smoothing parameter. Articles [35]-[36] use the cross-validation method to optimize the choice of parameters when the process is real-time. The objective of this work is to give a criterion for the selection of the smoothing parameters used in the estimation of the spectral density Φ. Let's note by $f(x_1,x_2) = (\Phi(x_1,x_2))^{\frac{q}{\alpha}}$ and $h = \frac{1}{M_n}$ the width of the two spectral windows. We are therefore looking for a criterion $CV(h)$ allowing us to select $h$to minimize the mean integrated square error (MISE), where

$$MISE(h) = \int\int E[f_n(x_1,x_2) - f(x_1,x_2)]^2 \rho(x_1,x_2) dx_1 dx_2, \qquad (5)$$

ρ being a weight function that is assumed to be known and zero outside of $[0,2\pi]\times[0,2\pi]$. Although MISE(h) is a suitable measure for the squared error of the estimator, it cannot choose h, since it itself depends on the unknown function f. We adopt the cross-validation method proposed in [35]. Indeed, consider the integrated square error (ISE) defined by:

$$ISE(h)=\int\int[f_n(x_1,x_2)-f(x_1,x_2)]^2\rho(x_1,x_2)dx_1dx_2=A-2C+B$$
$$whereA=\int_0^{2\pi}\int_0^{2\pi}f_n^2(x_1,x_2)\rho(x_1,x_2)dx_1dx_2$$
$$C=\int_0^{2\pi}\int_0^{2\pi}f_n(x_1,x_2)f(x_1,x_2)\rho(x_1,x_2)dx_1dx_2$$
$$B=\int_0^{2\pi}\int_0^{2\pi}f^2(x_1,x_2)\rho(x_1,x_2)dx_1dx_2.$$

B being independent of h, the choice h minimizing ISE(h) amounts to choosing the h minimizing A-2C. It is clear that the term A is computable since we know f_n, whereas, in the term C, contains f unknown. We use the principle of “leave-out- I”.

3. CONSTRUCTION OF THE CROSS-VALIDATION ESTIMATOR

Let $j,j'\in\{0,1,\dots,n-1\}$ such that $\frac{2\pi j}{n}\in U_n$ and $\frac{2\pi j\prime}{n}\in U_n$. The construction of “leave-out- I" consists of finding an estimator $f_n^{j,j'}(\omega_j,\omega_{j\prime})$ that replace $f(w_j,w_{j\prime})$ in the expression of C and such that $I_n(\omega_j,\omega_{j\prime})$ and $f_n^{j,j'}(\omega_j,\omega_{j\prime})$ are asymptotically independent. Thus, we can estimate C by: $\frac{1}{\bar{n}^2}\sum_{j\in A_n}\sum_{j'\in A_n}f_n^{j,j'}(w_j,w_{j'})I_n(w_j,w_{j'})\rho(w_j,w_{j'})$ where $w_j=\frac{2\pi j}{n},w_{j'}=\frac{2\pi j'}{n},\bar{n}=\left[\frac{n-1}{2}\right]$ and $A_n=\left\{j\in\{0,1,\dots,n-1\}such\ that\frac{2\pi j}{n}\in U_n\right\}$

$$f_n^{j,j'}(x_1,x_2)=\int_{U_n^2}I_n^{j,j\prime}(u_1,u_2)W_n(x_1-u_1,x_2-u_2)du_1du_2,\ where$$
$$I_n^{j,j'}(u_1,u_2)=I_n(u_1,u_2)if\ \ (u_1,u_2)\notin B_{j,j'}$$
$$I_n^{j,j'}(u_1,u_2)=\theta_1(u_1,u_2)I_n(w_{j-1},w_{j'-1})+\theta_2(u_1,u_2)I_n(w_{j+1},w_{j'-1})+\theta_3(u_1,u_2)I_n(w_{j-1},w_{j'+1})+\theta_4(u_1,u_2)I_n(w_{j+1},w_{j'+1})\qquad otherwise$$

$B_{j,j'}=]w_{j-1},w_{j+1}[\times]w_{j'-1},w_{j'+1}[$. The construction of $I_n^{j,j'}(u_1,u_2)$ where $(u_1,u_2)\in A_{j,j'}$ is done as if I_n was bi-linear. In this case,

$\theta_1(u_1,u_2)=\alpha\beta$; $\theta_2(u_1,u_2)=(1-\alpha)\beta$; $\theta_3(u_1,u_2)=\alpha(1-\beta)$ and $\theta_4(u_1,u_2)=(1-\alpha)(1-\beta)$ where $\alpha=\frac{u_1-w_{j+1}}{w_{j-1}-w_{j+1}}$ and $\beta=\frac{u_2-w_{j'+1}}{w_{j'-1}-w_{j'+1}}$.

The following proposition shows that $f_n^{j,j\prime}$ is an estimator asymptotically unbiased of the function f.

Theorem 3.1 *For all* $(x_1,x_2)\in Q_p^2$

$$E\left[f_n^{j,j'}(x_1,x_2)-f_n(x_1,x_2)\right]=O\left(\frac{1}{n^2}\right).$$

From this result, we establish our criterion, noted CV "cross validation" defined by:

$$CV(h) = \quad CV_1(h) + \int_{U_n{}^2} f^2(u_1,u_2)\rho(u_1,u_2)du_1du_2$$

$$where \;\; CV_1(h) = \int_{U_n{}^2} f_n^2(u_1,u_2)\rho(u_1,u_2)du_1du_2 -$$

$$\frac{2}{n^2}\sum_{j\in A_n}\sum_{j'\in A_n} f_n^{j,j'}(w_j,w_{j'})I_n(w_j,w_{j'})\rho(w_j,w_{j'})$$

The widths of spectral windows will be chosen at the points $\hat{h}$ minimizing the criterion $CV(h)$:

$$\hat{h}_1 = \underset{h}{\operatorname{argmin}} CV(h) = \underset{h}{\operatorname{argmin}} CV_1(h) \qquad (6)$$

To facilitate writing without losing generality, we consider:
$\rho(u_1,u_2) = \frac{1}{2\pi}$ on U_n^2 and zero outside.

We will show results similar to those given in [37]. This is to show that on average, the criterion CV(h_) and ISE(h) are asymptotically close and that the variance of CV(h) is asymptotically zero. Thus, the parameters hˆ minimizing the criterion CV(h) also minimize the integral squared error (ISE) when n is large enough. These results are stated in the following theorem.

Theorem 3.2 *We have*

$$|E\{CV(h) - ISE(h)\}| = O\left(\frac{1}{n^2}\right).$$

$$var\{CV(h)\} = O\left(\frac{1}{n^2h^2}\right)$$

Thus, since

$$E\{[CV(h) - MISE(h)]^2\} =$$

$$var\{CV(h)\} + [E\{CV(h) - MISE(h)\}]^2 = \quad O\left(\frac{1}{n^2h^2}\right).$$

The widths of the spectral windows $\hat{h}_1$ and $\hat{h}_2$ obtained by cross validation, defined in (6), are asymptotically optimal, i.e. the integrated square error at $\hat{h}$ converges in probability to the small integrated square error.

Theorem 3.3*The width of the spectral windows $\hat{h}$ obtained by cross validation are asymptotically optimal:*

$$\frac{ISE(\hat{h})}{ISE(\hat{\hat{h}})} \longrightarrow 1 \quad in\ probability, \quad where$$

$$\hat{h} = \underset{h}{\operatorname{argmin}} CV(h) \quad and \quad (\hat{\hat{h}}) = \underset{h}{\operatorname{argmin}} ISE(h).$$

To show this result, we use the similar technique used in [38].

4. Conclusion

The method proposed in this work consists in giving the smoothing parameter optimizing the estimate of the spectral density for an alpha-stable process in p-adic time. To achieve this, we used the cross-validation technique, which is well suited to this kind of situation. This work can be used in various fields of application to model phenomena whose variance is large enough such as:

- Drones take Dynamic Images. These images are often disturbed by climatic conditions. They can have a large variance and be modeled by a alpha stable random field.

- The presence of certain microorganisms in agricultural soil varies significantly can be considered as an alphastable random field.

This work will be extended to a more general case where the measure is mixed.

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