EXPONENTIAL WINDOW FAMILY

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ABSTRACT

In this paper we propose a new class of 2-parameter adjustable windows, namely Exponential window, based on the exponential function [1,2]. The Exponential window is derived in the same way as Kaiser window was derived, but our proposed window is more computationally efficient because in its time domain function it has no power series expansion. First, the spectrum design equations for the Exponential window are established, and the spectral comparisons are performed with Cosh, Kaiser and ultraspherical windows. The proposed window is compared with Cosh and Kaiser windows, and the results show that for the same window length and mainlobe width the Exponential window provides better sidelobe roll-off ratio characteristic, which may be important for some applications, but worse ripple ratio. The second comparison is performed with ultraspherical window for the same window length, mainlobe width and sidelobe roll-off ratio and the results demonstrate that the Exponential window exhibits better ripple ratio for the narrower mainlobe width and larger sidelobe roll-off ratio, but worse ripple ratio for the wider mainlobe width and smaller sidelobe roll-off ratio.

KEYWORDS

Window function, Exponential window, Cosh Window, Kaiser window, Ultraspherical window

1. INTRODUCTION

Providing new window functions (or simply as windows) is in interest, because they are widely used in digital signal processing applications, e.g., signal analysis and estimation, digital filter design and speech processing [1-3]. In literature many windows have been proposed [4-16]. Since the best window depends on the applications, they are known as suboptimal solutions.

Kaiser window [5] is a well-known two parameter flexible window and widely used for FIR filter design and spectrum analysis applications. It performs good results because it achieves close approximation to the discrete prolate spheroidal functions that have maximum energy concentration in the mainlobe. With adjusting its two independent parameters, the window length and the shape parameter, it can control the spectral parameters main lobe width and ripple ratio for various applications.

Sidelobe roll-off ratio, which is important for some applications, is another window spectral parameter to differentiate the performances of the windows. For beamforming applications, the higher sidelobe roll-off ratio means that it can reject far end interferences better [11]. For the design of nonrecursive digital filters, it reduces the far end attenuation for stopband energy [12], and reducing the energy leak from one band to another for speech processing [17].

In terms of roll-off ratio characteristic the Kaiser window provides better sidelobe than the other well-known two parameter adjustable windows such as Dolph-Chebyshev [4] and Saramaki [6].

Therefore, providing a window performing higher sidelobe roll-off characteristic than the Kaiser window will be useful for some signal processing applications.

In this paper, a new window based on the exponential function is proposed to provide higher sidelobe roll-off ratio than Kaiser window to be useful for some applications.

2. DERIVATION OF THE EXPONENTIAL WINDOW

In this section, a brief explanation about how to derive the proposed window function is given.

2.1. Windows

An N-length window, denoted by w(nT), is a time domain function which is nonzero for n≤⌊(N-1)/2⌋ and zero for otherwise. They are generally compared and classified in terms of their spectral characteristics. The frequency spectrum of $w(nT)$ can be found by

$$W(e^{jwT}) = |A(w)|e^{j\theta(w)} = w(0) + 2\sum_{n=1}^{(N-1)/2} w(nT)\cos wnT \qquad (1)$$

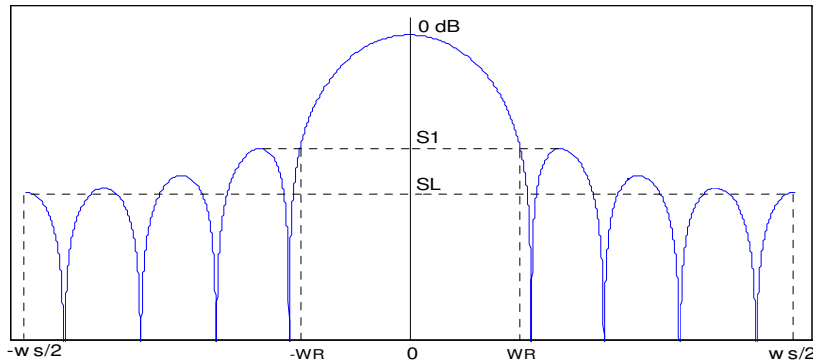where T is the sample period. A typical window has a normalized amplitude spectrum in dB range as in Figure 1.

Figure 1. A typical window's normalized amplitude spectrum

Normalized spectrum in Fig.1 can be obtained from

$$\left|W_N(e^{jwT})\right| = 20\log_{10}(|A(w)|/|A(w)|_{max}) \qquad (2)$$

The common spectral characteristic parameters to distinguish the windows performance are the mainlobe width (w_M), the ripple ratio (R) and the sidelobe roll-off ratio (S). From Figure 1, these parameters can be defined as

w_M = Two times half mainlobe width = $2w_R$
R = Maximum sidelobe amplitude in dB - Mainlobe amplitude in dB = S_1
S = Maximum sidelobe amplitude in dB - Minimum sidelobe amplitude in dB = S_1-S_L

In the applications, it is desired for a window to have a smaller ripple ratio and a narrower mainlobe width. But, this requirement is contradictory [3].

2.2. Kaiser Window

Kaiser window is defined in discrete time, as [3, 5]

$$w_k(n) = \begin{cases} \dfrac{I_0\left(\alpha_k\sqrt{1-\left(\dfrac{2n}{N-1}\right)^2}\right)}{I_0(\alpha_k)} & |n| \leq \dfrac{N-1}{2} \\ 0 & otherwise \end{cases} \qquad (3)$$

where α_k is the adjustable shape parameter, and $I_0(x)$ is the modified Bessel function of the first kind of order zero, which is described by the following power series expansion as

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2 \qquad (4)$$

While an approximation closed formula for the Kaiser window spectrum is defined [3], the exact Kaiser spectrum can be obtained from Eq. (1). Note that T=1 is considered as the normalization for the rest of paper.

As known from the fixed windows while the window length, N, increases the mainlobe width decreases but ripple ratio remains generally constant. And, larger values of the shape parameter, α_k, result in a wider mainlobe width and a smaller ripple ratio.

2.3. Exponential Window

From Figure 2, it can be seen that exp(x) and Io(x) have the same shape characteristic.

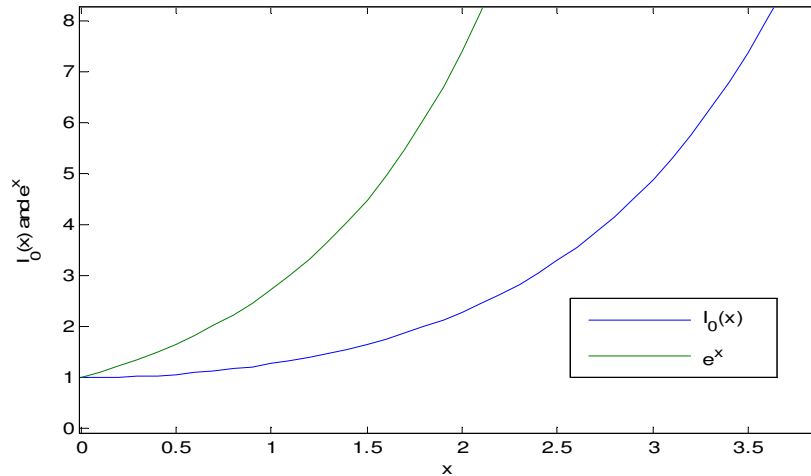Figure 2. The functions Io(x) and e^x

Therefore, a new window, called "*Exponential window*" for this paper, can be proposed as

$$w_e(n) = \begin{cases} \dfrac{e^{\left(\alpha_e\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right)}}{e^{\alpha_e}} & |n| \leq \dfrac{N-1}{2} \\ 0 & otherwise \end{cases} \qquad (5)$$

Like the Kaiser window, the Exponential window has two independent parameters, namely the window length (N) and the adjustable shape parameter (α_e). Figure 3 shows the time domain characteristic of the exponential window for various values of the parameter α_e with N = 51. It is seen that $\alpha_e = 0$ corresponds to the rectangular window as in the case for the Kaiser window. For larger values of α_e, the Exponential window becomes to have a Gaussian shape.
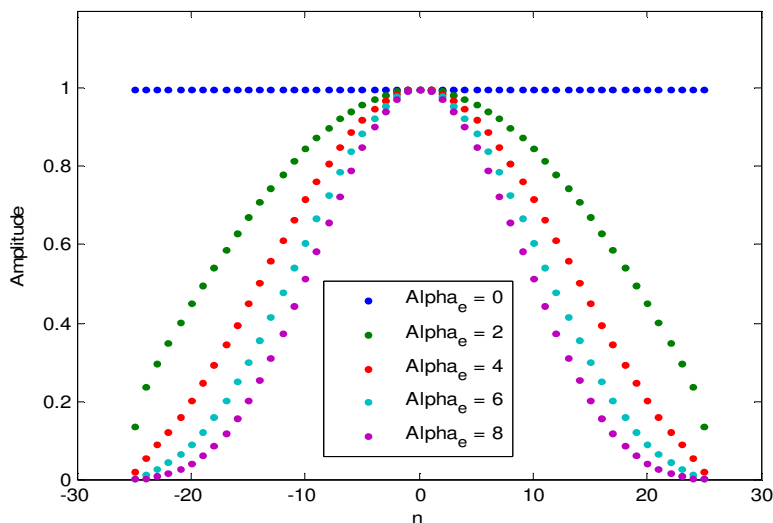

Figure 3. Exponential window in time domain for α_e = 0, 2, 4, 6, and 8 with N = 51

The exact spectrum for the Exponential window can be obtained from Eq. (1). Figure 4 shows the effect of α_e on the Exponential window spectrum for a fixed value of length N = 51. And, Table 1 summarizes the numerical data in Figure 4. As seen from the figure and table, an increase in α_e results in a wider mainlobe width and a smaller ripple ratio.
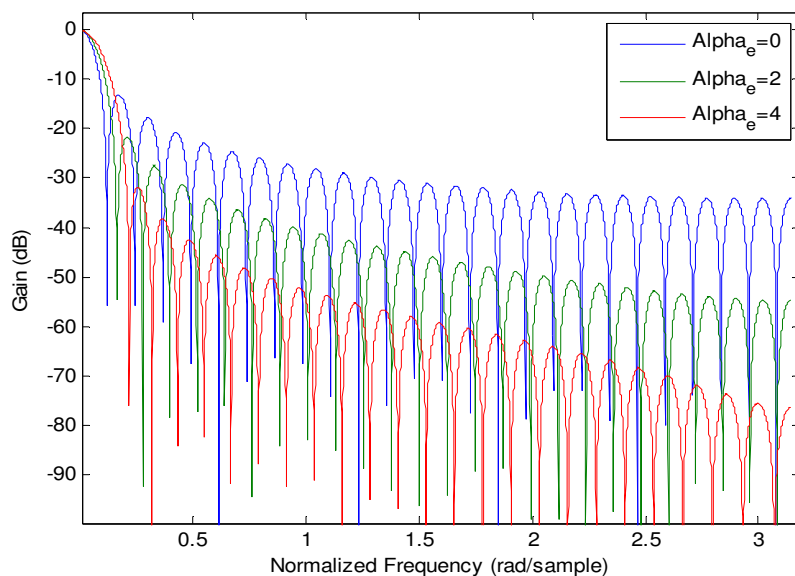

Figure 3. Proposed window spectrum in dB for α = 0, 2, and 4 and N=51

Table 1. Spectral data for Exponential window

Window	N	α	w_R	R	S
Proposed-1	51	0	0.1	-13.25	20.9
Proposed-2	51	2	0.15	-21.73	32.95
Proposed-3	51	4	0.21	-31.84	44.54

3. SPECTRUM DESIGN EQUATIONS

It is important for some applications such as the spectrum analysis to have the window design equations which define the window parameters in terms of the spectral parameters.

To obtain the spectrum design equations for the Exponential window, it is necessary to find the relations between the window parameters and spectral parameters empirically. Figure 4 shows the relation between α_e and the ripple ratio for the window lengths N = 51 and 101.
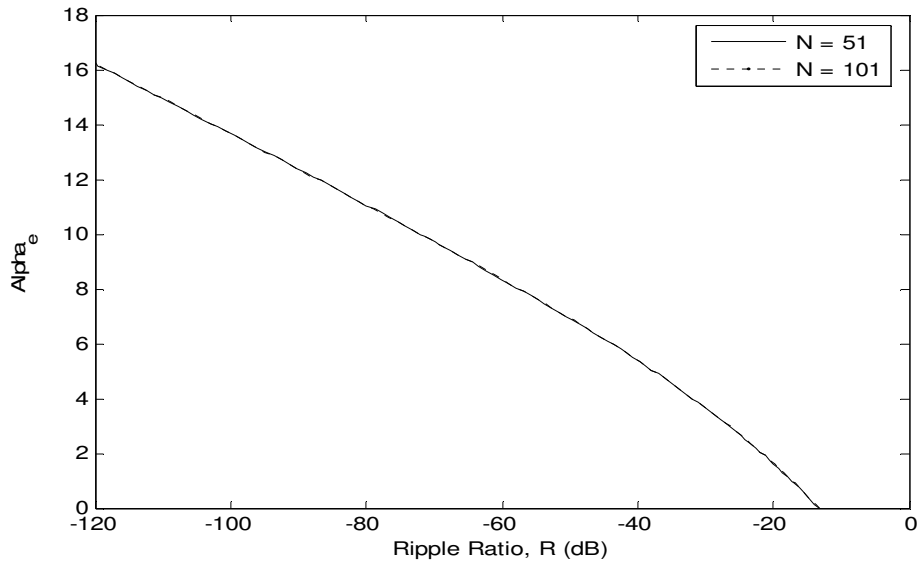

Figure 4. Relation between α_e and R for the Exponential window with N = 51 and 101

It is seen from Figure 4 that the window length parameter doesn't affect the relation between the adjustable parameter α_e and the ripple ratio. Therefore, using the curve fitting method in MATLAB, the first design equation for α_e in terms of the ripple ratio can be obtained as

$$\alpha_{e,Appr} = \begin{cases} 0 & R > -13.26 \\ -1.513x10^{-3}R^2 - 0.2809R - 3.398 & -50 < R \le -13.26 \\ -1.085x10^{-4}R^2 - 0.1506R - 0.304 & -120 \le R \le -50 \end{cases} \quad (6)$$

The quadratic approximation model given by Eq. (6) for the adjustable parameter α_e is plotted in Figure 5. It is seen that the proposed model provides a good approximation for N = 101. Moreover, the approximation error for the first design equation for N = 101 is plotted in Figure 6. It is observed that the amplitude of deviations in the alpha is lower than 0.06 which corresponds to very small error in the ripple ratio.

More accurate results can be obtained by restricting the range or using higher order approximations, but the proposed model for the Exponential window is adequate for most applications like the Kaiser model.
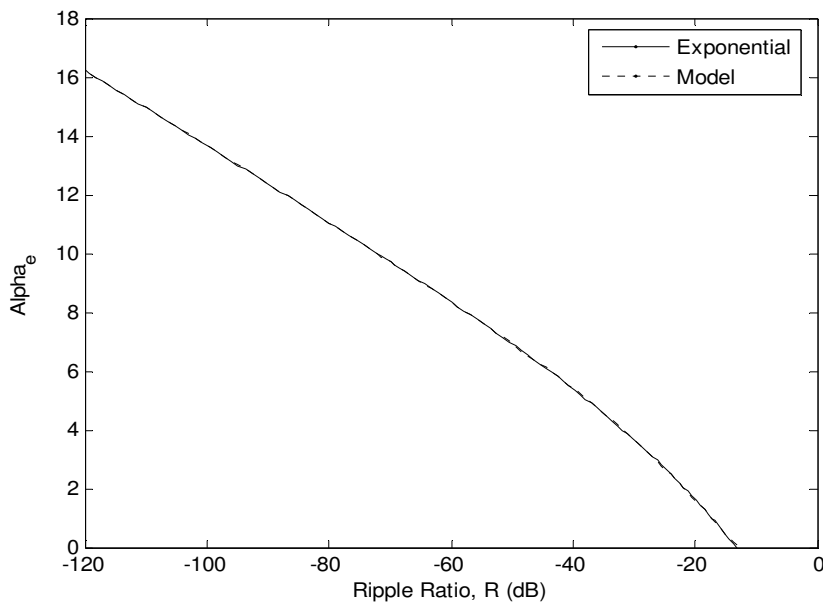

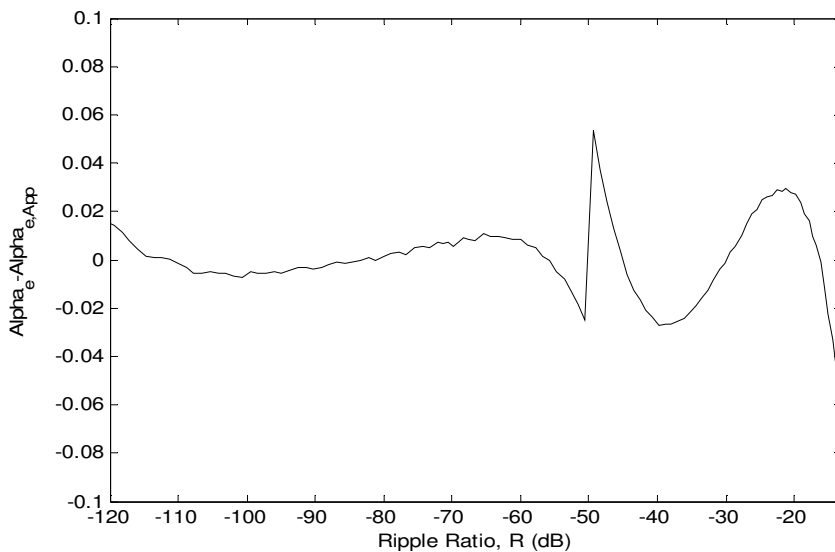Figure 5. Approximated model for α_e of the Exponential window with N = 101

Figure 6. Error curve of approximated α_e versus R for N = 101

The second design equation is the relation between the window length and the ripple ratio. To predict the window length for a given quantities R and w_R, the normalized width parameter $D_w = 2w_R(N-1)$ is used [11]. The relation between D_w and R for the Exponential window with N = 51 and 101 is plotted in Figure 7.
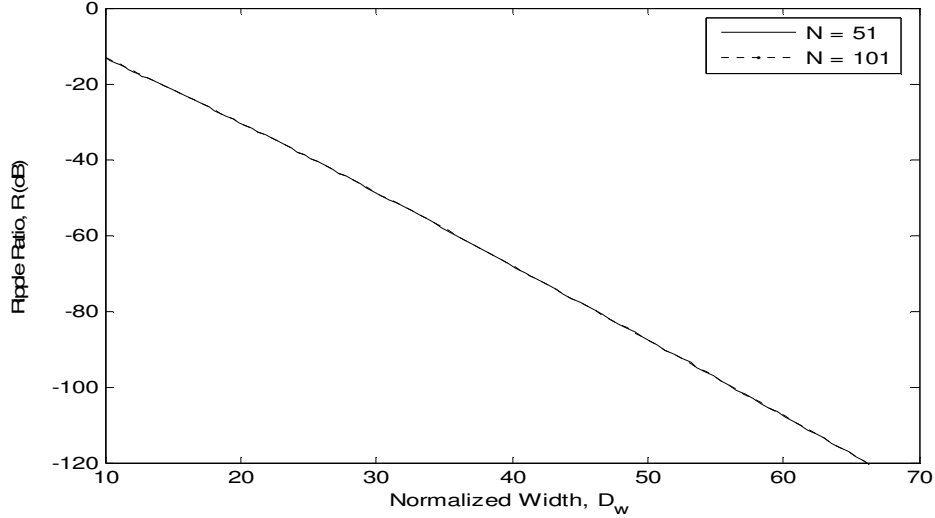
Figure 7. Relation between D_w and R for the Exponential window with N = 51 and 101

It is seen from Figure 7 that as the ripple ratio becomes smaller the mainlobe width becomes wider. Also, it is observed from the same figure that the window length has no effect on the relation between the ripple ratio and normalized mainlobe width. By using the curve fitting method, an approximate design relationship between the normalized width (D_w) and the ripple ratio (R) can be established as

$$D_{w,Appr} = \begin{cases} 0 & R > -13.26 \\ -7.58x10^{-5}R^3 + 7.22x10^{-3}R^2 - 0.3566R + 4.312 & -50 < R \leq -13.26 \\ -1.297x10^{-4}R^2 - 0.5281R + 4.708 & -120 \leq R \leq -50 \end{cases} \tag{7}$$

The approximation model given by Eq. (7) for the normalized mainlobe width is plotted in Figure 8. It is seen that the proposed model provides a good approximation for N = 101.
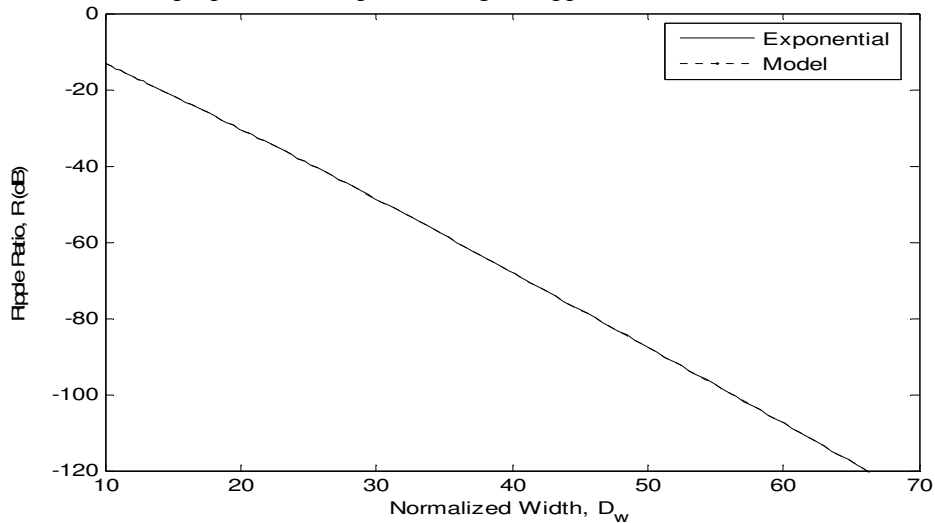

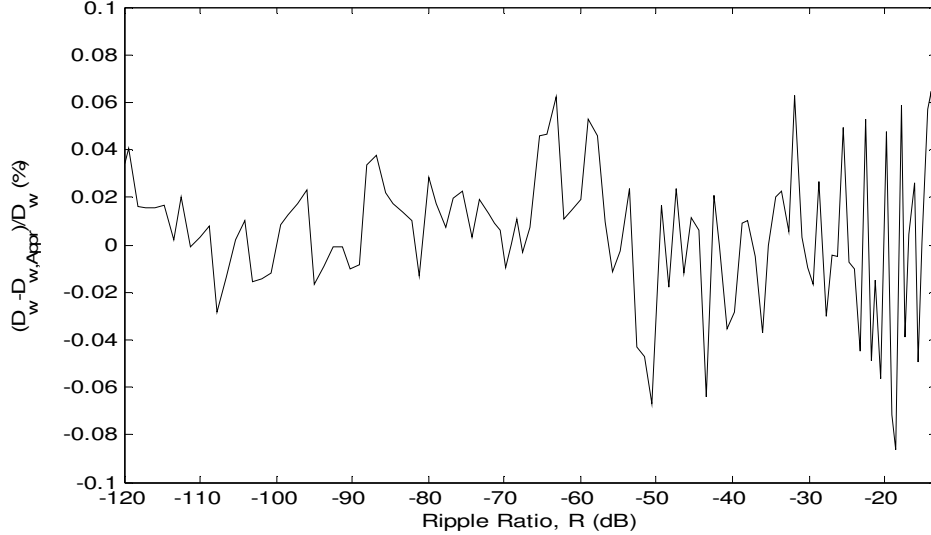Figure 8. Approximated model for D_w of the Exponential window with N = 101

Figure 9.   Relative error of approximated D_w for the Exponential window in percent versus R with N = 101

The relative error of approximated normalized width in percent versus the ripple ratio for N = 101 is plotted in Figure 9. The percentage error in the model changes between 0.065 and -0.086. This error range satisfies the error criterion in [11] which states that the predicted error in the normalized width must be smaller than 1 %.

An integer value of the window length N can be predicted from [11]

$$N \geq \frac{D_{w,Appr}}{2w_R} + 1 \tag{8}$$

Using the equations (6) through (8), an Exponential window can be designed for satisfying the given prescribed values of the ripple ratio and mainlobe width.

In some applications [17], larger sidelobe roll-off ratio may be desired. Figure 10 shows the change in the sidelobe-roll off ratio in terms of the normalized mainlobe width parameter for N = 51 and 101. From the figure it can be seen that the sidelobe roll off ratio becomes larger as normalized width increases until one of the sidelobes is dropped due to higher value of alpha. Unlike in the case of ripple ratio, a change in the window length affects significantly the sidelobe roll-off ratio characteristic of the Exponential window.
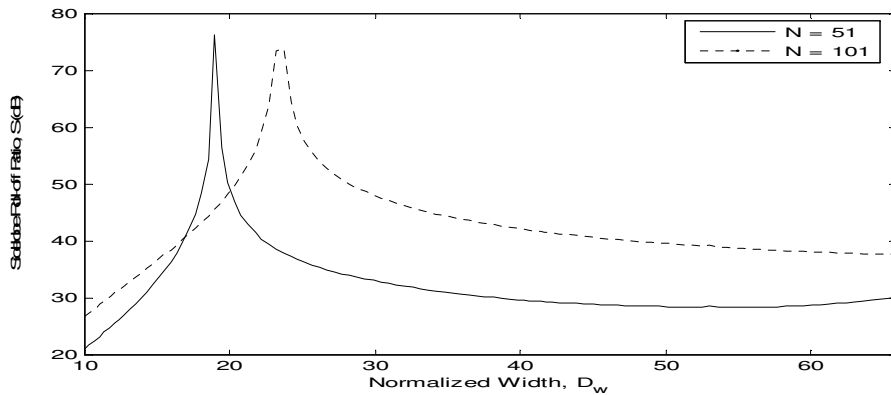

Figure 10. Relation between D_w and S for the Exponential window with N = 51 and 101

4. SPECTRUM COMPARISON EXAMPLES

4.1. Comparison with Kaiser and Cosh Windows

Figure 11 shows a general comparison of the Cosh window in a wide range with the Exponential and Kaiser windows in terms of the ripple ratio versus normalized mainlobe width for N = 101. The figure demonstrates that the Kaiser window provides smaller ripple ratio than the others for the same mainlobe width. For the range D_w < 25, the Cosh window produces smaller ripple ratio than the Exponential window. And, for the range 25 < D_w the Cosh and Exponential windows perform the same ripple ratio characteristic.
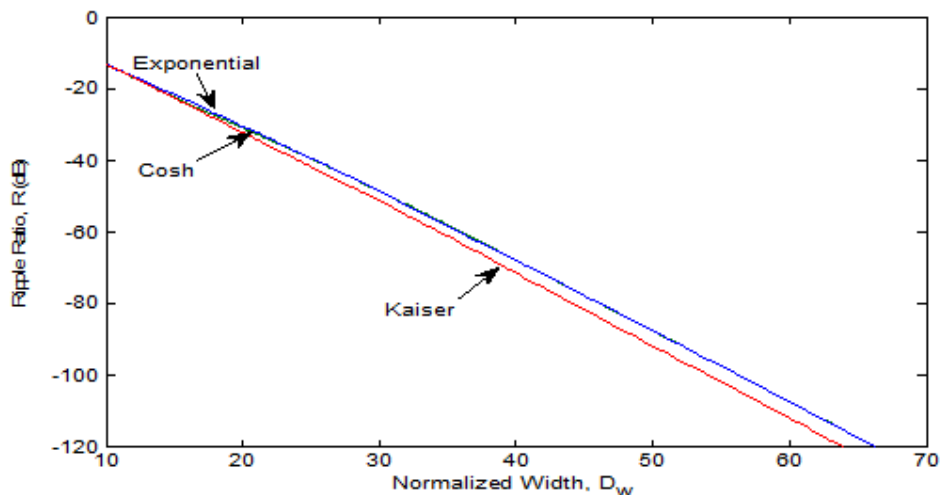

Figure 11. Ripple ratio comparison between the Cosh, Exponential and Kaiser windows
for N = 101

The simulation results for the sidelobe roll-off ratio comparison is given for N = 101 in Figure 12. It is seen that the Cosh window performs better than the Kaiser window but worse than the Exponential window in terms of the sidelobe roll-off ratio for the same mainlobe width until one sidelobe is lost where the peak values occur.
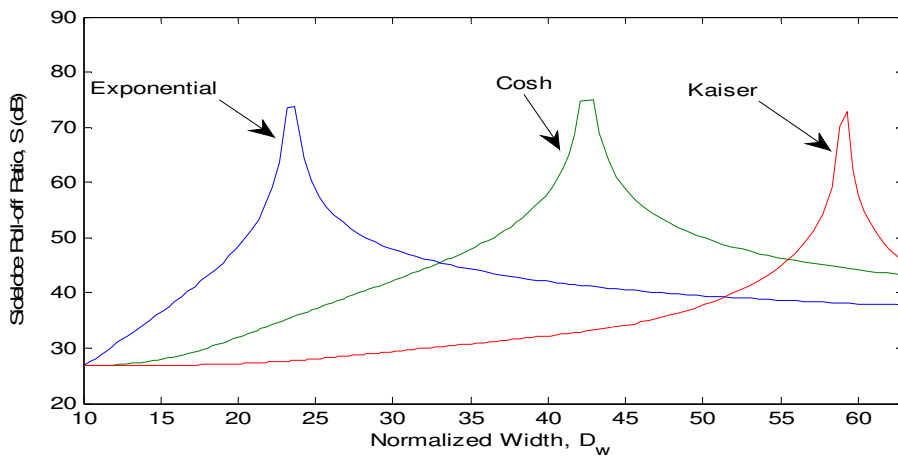

Figure 12. Sidelobe roll-off ratio comparison between the Cosh, Exponential
and Kaiser windows for N = 101

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4.2. Comparison with Ultraspherical Window

Two specific examples are given for the comparison between the Exponential and ultraspherical windows. The first comparison example is performed for the narrower mainlobe width and larger sidelobe roll-off ratio with N = 51.
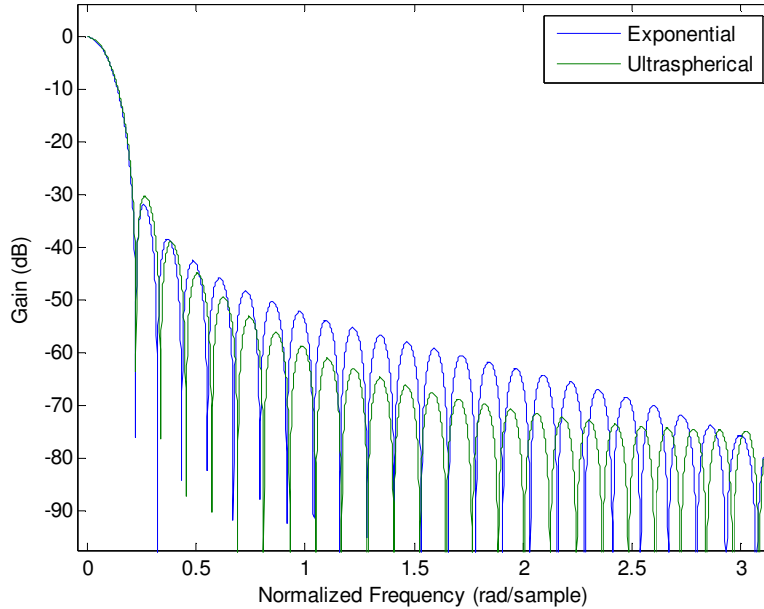

Figure 13. Comparison of the proposed and ultraspherical windows for narrower mainlobe width and larger sidelobe roll-off ratio for N=51

The simulation result given in Figure 13 and Table 2 which summarizes the figure shows that the three-parameter ultraspherical window provides a better ripple ratio than the Exponential window for the same window length, mainlobe width and sidelobe roll-off ratio. The ultraspherical window parameters for this example are $\mu = 1.99999$ and $x_\mu = 1.00039$.

Table 2. Data for the first comparison example

Window	N	w_R	S	R
Exponential	51	0.164	37.81	-24.1
Ultraspherical	51	0.164	37.81	-23.02

The second comparison example is given for the wider mainlobe width and smaller sidelobe roll-off ratio for N = 51. The simulation result given in Figure 14 and Table 3 shows that the Exponential window provides a better ripple ratio than the ultraspherical window in this case. The ultraspherical window parameters for this example are $\mu = 1.66635$ and $x_\mu = 1.00973$.
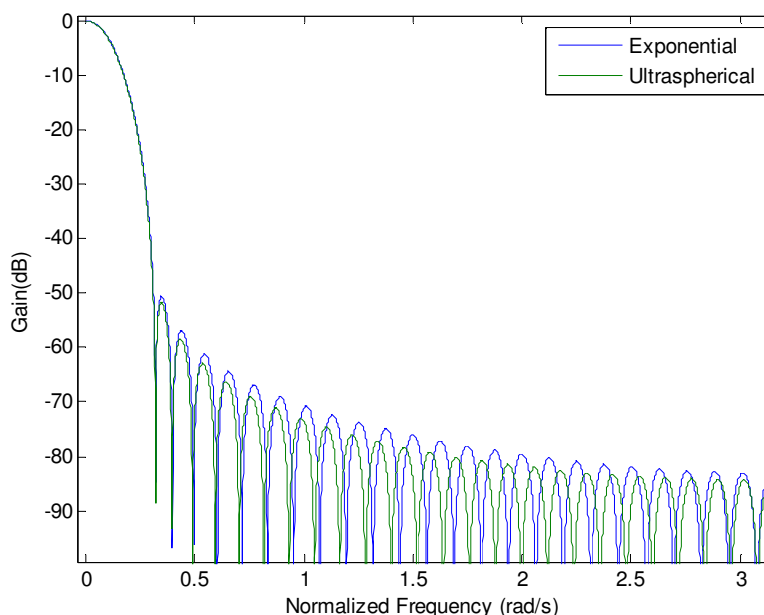
Figure 14. Comparison of the proposed and ultraspherical windows for wider mainlobe width and smaller sidelobe roll-off ratio for N=51

From Figures 13 and 14, the ripples between the maximum and the minimum sidelobe amplitudes can also be observed to be higher for the Exponential window.

Table 3. Data for the first comparison example

Window	N	w_R	S	R
Proposed	51	0.31	32.48	-50.53
Ultraspherical	51	0.31	32.48	-51.75

5. CONCLUSIONS

In this paper, a new 2-parameter window family based on the exponential function has been proposed. Since it's derived using the exponential function, it has been called "Exponential window" for this paper. First, the proposed window family has been introduced by giving its derivation and mathematical definition. And then, its spectrum design equations using curve fitting method in MATLAB have been obtained.

To demonstrate the performance of the proposed window, its spectral comparisons have been performed with Cosh, Kaiser and ultraspherical windows. Comparison with Cosh and Kaiser windows showed that the Exponential window provides better sidelobe roll-off ratio characteristic, but presents worse ripple ratio for the same window length and mainlobe width. As for the comparison with 3-paramater ultraspherical window, for the same the window length, mainlobe width and sidelobe roll-off ratio parameters, the Exponential window presents better ripple ratio for the narrower mainlobe width and larger sidelobe roll-off ratio, but exhibits worse ripple ratio for the wider mainlobe width and smaller roll-off ratio.

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